DEA with Linear Object Function

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Abstract: The classic DEA model maximizes the ratio of aggregate output over aggregate input. If we denote the aggregated output as revenues and the aggregated input as costs, then the DEA model works with the ratio of revenues and costs. In the article, instead of the ratio of output over input, the difference between revenues and costs, i.e. profit, is used. It is shown that the fractional objective function and the difference between revenues and costs may differ in the optimal solution. In the proposed DEA model, the objective function is profit, that is, the difference between revenues and costs, so it is a linear objective function. The proposed linear objective function DEA model is demonstrated by a numerical example and the differences in results are shown. This model is also analyzed and its possible modifications are shown.

Keywords: DEA model; DEA model with linear object function; DEA with linear-fractional objective function

JEL Classification: C44

1. Introduction in DEA Models

The DEA method was proposed to measure and compare the efficiency of production units based on the outputs and inputs of these units. It is based on the calculation and optimization of efficiency ratios, which is the ratio of aggregate output over aggregate input (hereafter efficiency index). The DEA model was published in the work (Farrell, 1957) "Measuring the efficiency of decision-making units" and by (Charnes et al., 1978), (Charnes & Cooper, 1962), and (Jablonský & Dlouhý, 2004), is also referred to as the CCR model.

The model is based on a certain number of outputs and a certain number of inputs, the aggregated output is the weighted sum of these outputs, and similarly the aggregated input is the weighted sum of the inputs. Input and output weights are model variables. The number of inputs and outputs for the evaluated production units is the same. The DEA method was designed to measure and compare the efficiency of production units based on the outputs and inputs of these units. It is based on the calculation and optimization of efficiency ratios, which is the ratio of aggregate output over aggregate input (hereafter efficiency index).

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Suppose we have r inputs and s outputs for each unit. There are n production units. Let's denote the inputs of the h-th production unit $x_{1,h}, x_{2,h},...x_{r,h}$, and the outputs of it $y_{1,h}, y_{2,h},.., y_{s,h}$. By aggregated input we mean the sum $\sum_{j=1}^{r} v_j x_{j,h}$, where $v_1, v_2, ..., vr$ are the weights of individual inputs. The aggregated output is the sum $\sum_{i=1}^{s} u_i y_{i,h}$ with the weights of the outputs $u_1, u_2, \dots, u_s$.

The aggregate efficiency index is then a share $I_h = \dfrac{\sum_{i=1}^{s} u_i \; y_{j,h}}{\sum_{j=1}^{r} v_j x_{j,h}}$.

The problem, however, is to determine the weights of inputs and outputs to maximized I_h; this can be done so that the values of I_k ($k = 1, 2, \dots, n$) of the aggregate efficiency index all production units are at most 1.

Mathematical model DEA (h-th production unit) is (1), (2), (3):

$$I_h = \frac{\sum_{i=1}^{s} u_i \; y_{j,h}}{\sum_{j=1}^{r} v_j x_{j,h}} \longrightarrow max \tag{1}$$

$$I_k = \frac{\sum_{i=1}^{s} u_i \; y_{j,k}}{\sum_{j=1}^{r} v_j x_{j,k}} \leq 1 \;, \qquad k = 1,2,\dots,n \tag{2}$$

$$u_i \geq 0, \;\; i = 1,2,\dots s, \;\; v_j \geq 0 \; j = 1,2,\dots,r \tag{3}$$

Optimal value of I_h is efficiency value of DEA method of h-th production unit. DEA efficiency value of all production units we get by solving n DEA models (1), (2), (3). Weights u and v of each production unit can be different, value of those gives maximal value of I_h.

If the value I_h of production unit reaches the value 1, it is an effective production unit. The order of the production units according to the degree of efficiency is given by the sizes of the corresponding maximal I_h values, which are in the interval (0, 1>.

2. Comparison of Optimal Solutions of the Optimization Model with **Profit Index** and Profit as Object Function

Analysis of the optimal solution of the problem with the objective function the ratio of revenues and costs (further profit index) and with the objective function profit as the difference between revenues and costs.

In this paragraph we will show the difference in optimal solutions of both models. First, we need to state some notation.

Notation:

X is the convex set of feasible solutions,

$f(X)>0$ the revenue of $X \in \boldsymbol{X}$,

$g(X)>0$ the costs of $X \in \boldsymbol{X}$,

profit $z(X)=f(X)-g(X)$,

profit index $I(X)= f(X)/g(X)$,

X_0 maximizes $I(X)$ on $\boldsymbol{X}$,

X' maximizes $z(X)$ on $\boldsymbol{X}$.

Proposition.

a) If $z(X') > 0$ then $g(X') \geq g(X_0)$ and $f(X') \geq f(X_0)$,

b) if $z(X') < 0$ then $g(X') \leq g(X_0)$ and $f(X') \leq f(X_0)$,

c) if $z(X') = 0$ then $I(X_0) = I(X') = 1$ and X_0 and X′ are optimal for both objective functions $z(X)$ and $I(X)$.

Providing $\frac{f(X_0)}{g(X_0)} > \frac{f(X')}{g(X')}$, it holds:

d) if $z(X') > 0$ then $g(X') > g(X_0)$ and $f(X') > f(X_0)$ (costs and revenue of X′ are higher than costs and revenue of X_0),

e) if $z(X') < 0$ then $g(X') < g(X_0)$ and $f(X') < f(X_0)$ (costs and revenue of X′ are lower than costs and revenue of X_0).

Proof.

Because X_0 maximizes the function $I(X)$ on X, so it holds

$$\frac{f(X_0)}{g(X_0)} \geq \frac{f(X')}{g(X')} \tag{6}$$

We easily see that from (6) follows next inequalities:

$\frac{f(X_0)}{g(X_0)} - 1 \geq \frac{f(X')}{g(X')} - 1$ and $\frac{f(X_0)-g(X_0)}{g(X_0)} \geq \frac{f(X')-g(X')}{g(X')} \frac{f(X_0)-g(X_0)}{g(X_0)} \geq \frac{f(X')-g(X')}{g(X')}$

$$f(X') - g(X') \geq f(X_0) - g(X_0) \geq \frac{g(X_0)}{g(X')}\big(f(X') - g(X')\big). \tag{7}$$

From (7) it follows two cases: a) and b).

Case a):

If $z(X') = f(X') - g(X') > 0$ then it follows from (7) by dividing it by $f(X') - g(X')$ the inequalities $\frac{g(X_0)}{g(X')} \leq 1$ and $g(X_0) \leq g(X')$ and

$$f(X') - g(X') \geq f(X_0) - g(X_0) \geq f(X_0) - g(X')$$

and finally, $f(X') \geq f(X_0)$.

Costs $g(X_0)$ at the solution X_0 are not higher than costs $g(X')$ at X‘ (they can be lower see case d)) and the same for revenue $f(X_0)$ and $f(X')$.

Case b):

If $z(x') = f(X') - g(X') < 0$ then from (7) it follows by dividing it by $z(X') = f(X') - g(X')$ that $\frac{g(X_0)}{g(x')} \geq 1$ and $g(X_0) \geq g(x')$ Then

$$\frac{f(X_0)}{g(X_0)} \geq \frac{f(X')}{g(X')} \geq \frac{f(X')}{g(X_0)}$$

hence $f(X') \geq f(X_0)$.

Costs $g(X_0)$ at the solution X_0 are not lower than costs $g(X')$ at X‘ and the same for revenue $f(X_0)$ and $f(X')$.

Case c) is trivial.

Cases d) and e) follow from a) and b) such that in (6) we assume a strong inequality.

QED.

The presented proposition shows that as a result of maximizing the efficiency index we can get a different solution than profit maximization. In that case and for $z(X') > 0$, the maximum profit can be higher than the profit corresponding to the model solution maximizing the index profit. Based on this, a linear DEA model can be proposed, which unlike the classic DEA model, will maximize profit, i.e. the difference between revenues and costs.

3. DEA Model with Linear Object Function

Object function (8) maximizes profit of *h*-th production unit provided that the profit of all production units (including this one) does not exceed the given parameter *H* (9).

In the model (8), (9), (10) (11), the parameter *H* is used, representing the maximum achievable profit, if *H = 0*, then this condition would coincide with the condition (2) of the classic DEA model. In case that *H = 0* the optimal solution is *v* = 0 and *u* = 0 for all production units and from it follows $\mathrm{p_h} = 0$. This can be prevented by condition (10).

$$p_h = \sum_{i=1}^{s} u_i \ y_{j,h} - \sum_{j=1}^{r} v_j x_{j,h} \longrightarrow max \tag{8}$$

$$p_k = \sum_{i=1}^{s} u_i \ y_{j,k} - \sum_{j=1}^{r} v_j x_{j,k} \le H , \qquad k = 1,2, \dots , n \tag{9}$$

$$\sum_{i=1}^{s} u_i + \sum_{j=1}^{r} v_j = 1 \tag{10}$$

$$u_i \ge 0, \ i = 1,2, \dots s, \ v_j \ge 0 \ j = 1,2, \dots , r \tag{11}$$

4. Example

The proposed linear DEA method is illustrated on the example of 13 production units, with three inputs (input1, input2 and input3) and two outputs (output1, output2). Input and output values are contained in Table 1. In Table 2 are the results of using the classic DEA method, linear DEA with value *H* = 0 and linear DEA with value *H* = 1,000. Two columns marked with DEA contains values I_h and the order of those values obtained by classical DEA model (1)-(3). Columns marked as *H* = 0 contains values p_h and the order of those values obtained by using linear DEA model (8)-(11) with *H* = 0. Similarly, the values in two columns labeled H = 1,000 are results of the linear DEA model (8)-(11) with the parameter *H* = 1,000.

Table 1. Example: set of production units

	input1	input2	input3	output1	output2
Z1	22.05	113.60	194.00	5,777	6
Z2	43.48	169.37	340.00	11,408	10
Z3	13.03	60.64	125	3,165	4
Z4	54.00	265.00	575.00	16,349	11
Z5	63.31	220.69	487.00	11,390	12
Z6	16.02	96.10	209.00	5,356	5
Z7	15.96	63.80	178.00	4,004	3
Z8	5.05	21.88	35.00	856	1
Z9	18.55	105.25	240.00	5,663	6
Z10	23.04	107.09	235.94	6,476	4
Z11	50.39	257.66	468.00	13,316	12
Z12	21.91	107.34	218.58	6,580	6
Z13	194.43	750.10	1,514.66	39,137	24

Table 2. Results of DEA and linear DEA

	DEA	DEA		*H = 0*	*H = 0*		*H = 1,000*	*H = 1,000*
	index of profit	Order		Profit	Order		Profit	Order
Z1	1	1-7		0	1-4		810.79	3
Z2	1	1-7		0	1-4		1,000	1-2
Z3	1	1-7		0	1-4		621.38	5
Z4	1	1-7		-16.387	5		140.13	7
Z5	0.868	12		-106.84	9		434.44	6
Z6	1	1-7		-41.37	6		701.53	4
Z7	0.9416	9		0	1-4		1,000	1-2
Z8	0.9045	11		-104.997	8		-95.6169	9
Z9	1	1-7		-107.063	10		-97.029	10
Z10	0.9612	8		-124.867	11		-113.866	11
Z11	0.9169	10		-177.734	12		-162.842	12
Z12	1	1-7		-34.898	7		-31.1432	8
Z13	0.774	13		-218.35	13		-198.081	13

In Table 2, differences in the order of the object function values of three modifications of DEA (the classical DEA, linear DEA with H = 0, linear DEA with H = 1,000) can be observed. For example, in classical DEA model there are 7 effective production units with value Ih = 0, while in the linear DEA model with H = 0 only 4 production units (value ph = 0) and for H = 1,000 two production units are effective (ph = 1,000). Production unit Z9 is effective by using classical DEA, but is not effective in the linear DEA models. Production unit Z13 is not effective in all three modifications DEA model.

The order of production units at three modifications is different, however their differences are not significant. It should be noted that findings from the example cannot be proven in general.

5. Conclusion

The article presents an alternative DEA method for measuring the efficiency of production units. It is based on the measurement of the efficiency of profit sizes, in contrast to the classic DEA method, which uses the ratio of revenues and costs (profit rates) for this measurement. The accompanying example shows the results of using both methods and compares the resulting requirements for the efficiency of the production units.

Conflict of interest: none.

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