Two-stage Serial Data Envelopment Analysis Models: Comparison of Approaches

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Abstract: Data envelopment analysis (DEA) is a modelling tool for assessment relative efficiency and performance of the set of homogeneous decision making units (DMUs) that transform multiple inputs into multiple outputs. Traditional models consider one-stage transformation – DMUs are black boxes that use multiple outputs and produce multiple inputs. In the contrary, network DEA models assume production process in a more general and complex way. In two-stage serial DEA models, the production process consists of two stages. The inputs of the first stage are used for production of the first stage outputs. These outputs enter the second stage as inputs and are used for production of the final outputs of the production process. The aim of this paper is to compare the most important approaches for evaluation of efficiency of the two-stage serial production processes based on the methodology of DEA. The properties of the models are discussed. A numerical example illustrates the results of all models.

Keywords: data envelopment analysis; network models; ranking; efficiency

JEL Classification: C44

1. Introduction

Traditional DEA models deal with efficiency analysis of one-stage production process, i.e. they analyze the relative efficiency of the transformation of multiple inputs into multiple outputs. The result of this analysis is an efficiency score that express that the unit under evaluation works on efficient frontier (is efficient) or not (is inefficient). This score is computed relatively to the other units of the homogeneous set of DMUs, i.e. adding or removing one unit of the set may (but need not) change the efficiency scores of other units. In general, the production process may be much more complex and cannot be expressed as one-stage process.

Network production processes may consist several interconnected sub-processes. Their efficiency may be analyzed by network DEA models. The simplest case of network structure is a two-stage serial process as presented in Figure 1. Let us consider the DMU$_i$ and denote the inputs of the first stage as x_{ij}, $i = 1,\dots,n$, $j = 1,\dots,m$, and the outputs of this stage that enter the second stage of the process as its inputs as z_{il}, $i = 1,\dots,n$, $l = 1,\dots,p$, where n is the number of DMUs, m is the number of the inputs of the first stage, and p is the number of the outputs of the second stage entering the next stage as inputs. The final outputs (t is their number) of the first stage (not entering the second stage) are y'_{ig} $i = 1,\dots,n$, $g = 1,\dots,t$. Similarly, the independent inputs (their number is s) of the second stage are y''_{ih}, $i = 1,\dots,n$, $h = 1,\dots,s$. The final outputs of the second stage and final outputs of the whole production process are denoted as y_{ik}, $i = 1,\dots,n$, $k = 1,\dots,r$, where r is their total number.

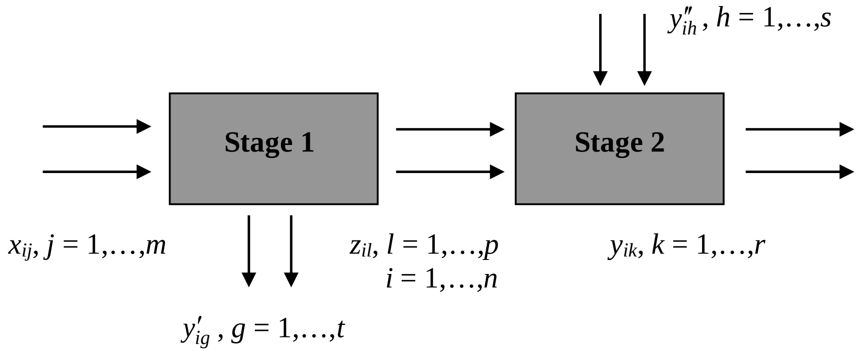

Figure 1: Two-stage serial production process

Network DEA models are of an increasing interest of researchers since the pioneering work (Färe & Grosskopf, 2000) was published. Tone and Tsutsui (2009) extended the current network models by measuring the efficiency using slacks and formulated their slack-based network model. The idea of slack-based measure (SBM) of efficiency was further developed by (Jablonský, 2018). The models for two-stage serial system were introduced in (Kao & Hwang, 2008) and (Chen et al., 2009). These models will be discussed in detail in the next section of the paper. An interesting approach for analysis of network production processes was published in (Mahdiloo et al., 2016). An extensive review of network DEA models can be found in (Kao, 2014).

The aim of this study is to compare the current modelling approaches for efficiency analysis of two-stage serial production processes. Section 2 presents basic definitions and formulations of two-stage serial DEA models. The main shortcoming of these models consists in their results. No DMU of the set need not be identified as efficient, i.e. of units may be inefficient. The model formulated in (Despotis et al., 2023) tries to overcome this shortcoming. This model is introduced in the last part of Section 2. The results of all presented models will be compared in Section 3 using an example of 24 insurance companies. The final section of the paper concludes the research and discusses its results.

2. Methodology

The history of DEA models started by publication of the paper (Charnes et. al., 1978). Their model considers multiple inputs being transformed by the DMU into multiple outputs is often known as CCR model. The input-oriented formulation of this model in its envelopment form can be written as follows:

Minimize θ_q^{CCR}

subject to $\sum_{i=1}^{n} x_{ij}\lambda_i + s_j^- = \theta_q x_{qj}$, $j = 1, ..., m,$ (1)

$\sum_{i=1}^{n} y_{ik}\lambda_i - s_k^+ = y_{qk}$, $k = 1, ..., r,$

$\lambda_i \geq 0,\ s_j^- \geq 0,\ s_k^+ \geq 0,\ i = 1,\dots, n, j = 1,..., m, k = 1,..., r,$

where s_j^-, j = 1,..., m, s_k^+, k = 1,..., r, are slack variables, and q is the index of the unit under evaluation. λ_i, i = 1, …, n are non-negative variables of the model that express the intensity of the units of the set in evaluation of the DMU$_q$. θ_q is another variable of the model that express the efficiency score of the DMU$_q$. Its maximum value 1 indicates that the unit q works in an efficient way, values less than 1 indicate inefficiency.

Two-stage serial models are much complicated that the traditional CCR model formulated above. The problem is that the level of efficiency of the unit under evaluation depends not only on the inputs and outputs of one stage. Here, increasing of efficiency of the first stage by increasing its outputs leads to decreasing of the efficiency of the second stage because of its higher inputs. Further in this and following sessions we consider the simplest case of the two-stage serial process where both stages have no independent variables – inputs of the first stage and outputs of the second stage, i.e. s = t = 0.

Kao and Hwang (2008) model connects both stages using the middle constraint in the formulation below and considering λ_i and μ_i, i = 1, …, n intensity variables for the first and second stages respectively. Its input-oriented formulation follows:

Minimize θ_q

subject to

$$\begin{aligned}
&\sum_{i=1}^{n} x_{ij}\lambda_i \le \theta_q x_{qj}, && j = 1,2,\dots,m, \\
&\sum_{i=1}^{n} z_{il}\lambda_i - \sum_{i=1}^{n} z_{il}\mu_i \ge 0, && l = 1,2,\dots,p, \\
&\sum_{i=1}^{n} y_{ik}\mu_i \ge y_{qk}, && k = 1,2,\dots,r, \\
&\lambda_i \ge 0,\ \mu_i \ge 0, && i = 1,2,\dots,n.
\end{aligned} \tag{2}$$

The efficiency score of the unit under evaluation θ_q is less or equal to 1. The maximum value 1 is reached for the unites that are efficient in both individual stages. Target input and output values can be derived as a linear combination of all other units of the set using optimal values of intensity variables λ_i and μ_i. The output-oriented formulation may be formulated in a similar way.

Chen et al. (2009) formulated two-stage serial DEA model with the assumption of constant returns to scale as follows:

Minimize $\theta_q - \phi_q$

subject to $\sum_{i=1}^{n} x_{ij}\lambda_i \le \theta_q x_{qj}, \quad j = 1,2,\dots,m,$ (3)

$$\sum_{i=1}^{n} z_{il}\lambda_i \geq \tilde{z}_{ql}, \qquad l = 1,2,\ldots,p,$$

$$\sum_{i=1}^{n} z_{il}\mu_i \leq \tilde{z}_{ql}, \qquad l = 1,2,\ldots,p,$$

$$\sum_{i=1}^{n} y_{ik}\mu_i \geq \varphi_q y_{qk}, \qquad k = 1,2,\ldots,r,$$

$$\theta_q \leq 1, \varphi_q \geq 1,$$

$$\lambda_i \geq 0, \mu_i \geq 0, \qquad i = 1,2,\ldots,n.$$

Similarly to the previous model (2), λ_i and μ_i, $i = 1, \ldots, n$ are intensity variables for both stages, θ_q and ϕ_q are input-oriented efficiency scores for the first stage and output-oriented scores for the second stage. New variables $\tilde{z}_{ql}$ of model (3) connect both stages of the production process. The DMU$_q$ is efficient in model (3) if it is efficient in both stages, i.e. $\theta_q = 1$ and $\phi_q = 1$. The inefficient units in the first stage have $\theta_q < 1$, the inefficiency in the second stage is indicated by $\phi_q > 1$. The final efficiency score of the DMU$_q$ may be derived as a product of both efficiency scores where ϕ_q must be considered as its reciprocal value. The problem of both models (2) and (3) is the possible inefficiency of all units of the set, i.e. no unit is efficient in both stages, which is a strange conclusion.

Jablonský (2018) combines model (3) and the SBM model introduced by Tone (2001) for the analysis of two-stage processes. This model measures the level of efficiency using slack variables, and its formulation is as follows:

Minimize
$$\psi_q = \frac{1 - \frac{1}{m}\sum_{j=1}^{m}\left(s_j^- / x_{qj}\right)}{1 + \frac{1}{r}\sum_{k=1}^{r}\left(s_k^+ / y_{qk}\right)}$$

subject to
$$\sum_{i=1}^{n} x_{ij}\lambda_i + s_j^- = x_{qj}, \qquad j = 1, \ldots, m, \tag{4}$$

$$\sum_{i=1}^{n} z_{il}\lambda_i \geq \tilde{z}_{ql}, \qquad l = 1,\ldots,p,$$

$$\sum_{i=1}^{n} z_{il}\mu_i \leq \tilde{z}_{ql}, \qquad l = 1,\ldots,p,$$

$$\sum_{i=1}^{n} y_{ik}\mu_i + s_k^+ = y_{qk}, \qquad k = 1, \ldots, r,$$

$$(1-\tau)z_{ql} \leq \tilde{z}_{ql} \leq (1+\tau)z_{ql}, \qquad l = 1,\ldots,p,$$

$$\lambda_i \geq 0,\ \mu_i \geq 0, \qquad i = 1, \ldots, n,$$

$$s_k^+ \geq 0, \quad k = 1,\ldots, r,$$

$$s_j^- \geq 0, \quad j = 1,\ldots, m,$$

In model (4), $s_j^-, j = 1,...,m$, and $s_k^+, k = 1,...,r$, are slack variables assigned to the j-th input and k-th output, respectively, τ is a parameter that express a maximum deviations of intermediate target values z_{ql} and new $\tilde{z}_{ql}$ variables. Objective function of model (4) is defined as a ratio of average slacks in the input space and average slacks in the output space – in a similar way as in SBM model (Tone, 2001). Model (4) is not linear in the objective function but can be transformed into a linear model quite easily. The DMUq is efficient in model (4) if all input and output slacks equal to 0, i.e. the objective function of model (4) equals to 1. Lower values indicate lower level of efficiency (or higher level of inefficiency).

A shortcoming of all presented models is a possible inefficiency status of all units, i.e. no DMU is found as efficient which is a strange result. Despotis et al. (2023) developed a simple approach that overcomes this shortcoming. The two-stage production process is considered in two perspectives:

- Perspective 1. The inputs of the first stage are considered to produce both outputs of the first stage and the final outputs of the second stage. In this perspective, the total number of inputs is m, and the number of outputs is (p + r). The efficiency score within this perspective can be derived by traditional CCR model (2) or by any other single stage DEA model. Let us denote θ^1 the score derived in this way.
- Perspective 2. The inputs of the first stage are taken together with the inputs of the second stage as the inputs of the new model that produce the final outputs. In this case, the total number of inputs is (m + p) and the number of outputs is r. The efficiency score of the new model can be derived by standard CCR model as in the previous case. Let us denote θ^2 the efficiency score in perspective 2.

The overall efficiency score of the two-stage serial system is defined as a geometric average of both scores θ^1 and θ^2. The authors of this approach prove that the application of this procedure leads to the result that at least one unit is overall efficient – (Despotis et al., 2023). If more than one unit is overall efficient, one of the super-efficiency models can be applied to discriminate among them. The inefficient unit can be ranked according to the values of their overall efficiency scores.

3. Results

The results of several modelling approaches for efficiency evaluation of two-stage serial processes will be compared with a data set of 24 insurance companies. The data set is not presented here but can be found in (Kao and Hwang, 2008). This model contains two production stages. The first one considers two inputs (operation expenses of the company and insurance expenses) and two outputs (direct written premiums and reinsurance premiums). Both outputs of the first stage are used as inputs of the second stage. The outputs of the second stage (underwritten profit and investment profit) are the final outputs of the whole system. The first stage evaluates marketing efficiency while the second stage is focused on profit efficiency.

Table 1 presents the results of the evaluation of efficiency of insurance companies using traditional radial one-stage models under the assumption of constant returns to scale – model (1). Table 1 contains the following results:

- The number of the DMU (insurance company).
- CCR 1 – CCR efficiency score of the first stage and ranking of the units according to this indicator. Five units (1, 12, 15, 19, and 24) are identified as efficient in the first stage.
- CCR 2 – CCR efficiency score of the second stage and ranking of the units. Four units (3, 5, 17, and 22) are efficient in the second stage.
- Geom – Geometric mean of the efficiency scores of both stages and ranking of DMUs. The results show that there is no unit efficient in both stages. It is interesting that the best unit according to the geometric mean is inefficient in both stages.
- CCR XY – the results of the CCR model that does not consider intermediate variables – inputs of the first stage are considered for production of final outputs only. Four units (2, 5, 12, and 22) are recognized as efficient in this case. The unit 24 that is efficient in the first stage but extremely inefficient in the second stage has the worse efficiency score among all units.

Table 1. Results of traditional models (constant returns to scale)

DMUs	CCR 1	Rank	CCR 2	Rank	Geom	Rank	CCR XY	Rank
1	0.9926	7	0.7134	7	0.8415	5	0.9840	6
2	0.9985	6	0.6275	10	0.7916	8	1.0000	1
3	0.6900	23	1.0000	1	0.8307	7	0.9884	5
4	0.7243	21	0.4323	16	0.5596	20	0.4882	14
5	0.8375	13	1.0000	1	0.9152	2	1.0000	1
6	0.9637	8	0.4057	18	0.6253	15	0.5938	13
7	0.7521	16	0.5378	13	0.6360	14	0.4698	16
8	0.7256	19	0.5113	15	0.6091	17	0.4148	19
9	1.0000	1	0.2920	23	0.5404	22	0.3270	22
10	0.8615	11	0.6736	9	0.7618	10	0.7807	10
11	0.7405	18	0.3267	22	0.4919	23	0.2826	23
12	1.0000	1	0.7596	6	0.8716	3	1.0000	1
13	0.8107	14	0.5435	12	0.6638	12	0.3527	20
14	0.7246	20	0.5178	14	0.6125	16	0.4696	17
15	1.0000	1	0.7047	8	0.8395	6	0.9793	7
16	0.9072	10	0.3847	19	0.5908	18	0.4717	15
17	0.7233	22	1.0000	1	0.8505	4	0.6349	11
18	0.7935	15	0.3737	20	0.5445	21	0.4271	18
19	1.0000	1	0.4158	17	0.6448	13	0.8220	9
20	0.9332	9	0.9014	5	0.9172	1	0.9351	8
21	0.7505	17	0.2795	24	0.4580	24	0.3328	21
22	0.5895	24	1.0000	1	0.7678	9	1.0000	1
23	0.8501	12	0.5599	11	0.6899	11	0.5990	12
24	1.0000	1	0.3351	21	0.5789	19	0.2571	24

Table 2 contains similar results as Table 1 but for the two-stage methods including the newest approach (Despotis et al., 2023). Table 2 has the same structure as the previous table.

There are presented efficiency scores derived by the methods and the ranking of DMUs according to these scores. The following methods are included:

- Kao – Kao and Hwang (2008) method – model (2). One can notice that no unit is efficient according to this approach, and all efficiency scores (except the DMU_9) are very low.
- Chen – The efficiency scores given by Che et al. (2009) method – model (3) – are higher that those from the previous case but again, no unit is efficient. The rankings of units obtained by both methods are the same.
- SBM – SBM model (4) is based on different principles than the remaining radial models. This is the reason that the ranking of units is here little different to the rankings of other methods.
- Despotis – The results of Despotis et al. (2023) model are presented in the last two columns of Table 2. They show the main property of this model that at lleast one unit is efficient – in our case three units (9, 10, and 12) are efficient.

Table 2. Results of two-stage models (constant returns to scale)

DMUs	Kao	Rank	Chen	Rank	SBM	Rank	Despotis	Rank
1	0.3936	4	0.6274	4	0.3578	2	0.9653	5
2	0.1472	21	0.3836	21	0.1777	17	0.4743	22
3	0.1738	17	0.4169	17	0.2010	15	0.5521	19
4	0.1714	18	0.4140	18	0.1268	20	0.7898	11
5	0.1317	22	0.3629	22	0.2901	9	0.5307	20
6	0.3530	7	0.5942	7	0.2550	11	0.8841	8
7	0.2200	14	0.4691	14	0.1736	18	0.6177	17
8	0.1640	20	0.4049	20	0.3045	7	0.6276	16
9	0.9338	1	0.9663	1	0.3378	3	1.0000	1
10	0.3962	3	0.6294	3	0.3338	4	1.0000	1
11	0.1644	19	0.4055	19	0.1842	16	0.5195	21
12	0.4774	2	0.6909	2	0.3259	5	1.0000	1
13	0.2779	10	0.5271	10	0.2972	8	0.7734	12
14	0.3059	9	0.5531	9	0.3064	6	0.8647	9
15	0.3498	8	0.5915	8	0.2200	13	0.9199	7
16	0.2760	12	0.5254	12	0.0793	22	0.8254	10
17	0.3745	6	0.6119	6	0.1116	21	0.9632	6
18	0.2778	11	0.5271	11	0.2615	10	0.7638	14
19	0.3748	5	0.6122	5	0.1498	19	0.9935	4
20	0.2598	13	0.5097	13	0.5430	1	0.7683	13
21	0.1998	15	0.4470	15	0.2066	14	0.6471	15
22	0.1995	16	0.4466	16	0.2321	12	0.6020	18
23	0.3936	4	0.6274	4	0.3578	2	0.9653	5
24	0.1472	21	0.3836	21	0.1777	17	0.4743	22

Correlation coefficients between all pairs of efficiency scores computed by the presented approaches are presented in Table 3. They show that the level of correlation between traditional efficiency measures computed by CCR model in the first and second stages are very low and the correlation is rather negative. This conclusion is not surprising as the outputs of the first stage lead to the lower efficiency of the second stage. The same holds for comparison of the measures derived by all other models – the level of correlation between efficiency scores of the first stage and all other measures is very low. The results of Despotis et a. (2023) model are strongly correlated with other two-stage models, and also with results of the model that does not take into account the intermediate variables (CCR XY). There is almost perfect positive correlation between Kao and Hwang (2008) and Chen et al. (2009) models.

Table 3. Correlation between the efficiency scores

	CCR 1	CCR 2	CCRXY	Geom	Despotis	Kao	Chen	SBM
CCR 1	1.0000							
CCR 2	-0.2274	1.0000						
CCR XY	0.2058	0.7828	1.0000					
Geom	0.1889	0.9026	0.8788	1.0000				
Despotis	0.2236	0.8428	0.9523	0.9375	1.0000			
Kao	0.2153	0.7691	0.9657	0.8735	0.9081	1.0000		
Chen	0.2002	0.7721	0.9737	0.8736	0.9148	0.9944	1.0000	
SBM	0.0278	0.7275	0.5993	0.7122	0.6669	0.5725	0.5757	1.0000

4. Discussion and Conclusions

DEA models for efficiency and performance evaluation of two-stage serial production processes are the simplest models of the broader family of network DEA models. The general production system can be considered as a combination of serial and parallel sub-processes. To evaluate the efficiency of such complex processes, various DEA based models have been formulated in the past. A general SBM model was formulated in (Tone & Tsutsui, 2009). A review of DEA network models was published in (Kao, 2014). This study formulated a set of the most often used two-stage serial DEA models and compared their results on a case of 24 insurance companies. The results show that the final efficiency score of the whole system depends mainly of the efficiency of the second stage. The results of all two-stage models considered in the study are more or less consistent with each other. Future research in this field will be focused on considering a dynamic factor to the analysis, i.e. formulation of dynamic DEA models because time factor is not solved satisfactorily yet.

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