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Vehicle Routing Problem with Choice of Nodes

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Abstract: Vehicle Routing Problem (VRP) consist in the problem of delivery of the items from depot to the other nods of the communication network. Nodes represent the recipients of the item delivery, items are transported by vehicles or any other means of transport. In a lot of applications of this problem, the requirements for the transport of items from depot come continually in time and they allow to delay the delivery. Therefore, there is no need to deliver this packet immediately at the time the requirement arrives. In this case (except for searching for the optimal routes) we need to decide which nodes will be in these routes and which not. The optimization will be related to the choice of nodes and minimization of routes containing these nodes. The objective function is the average length of the route to a unit of transported items. It is a linear-fractional function. In the article there are, except for the model with this objective function, suggested alternative methods including heuristic methods. Everything is illustrated on the numerical example.

Keywords: vehicle routing problem; integer programming; linear-fractional object function

JEL Classification: C44

1. Introduction

At the standard VRP there is given a set of nodes, a matrix of distances and requirements on the transfer in nodes (Laporte, 1992; Braysy & Gendreau, 2005). The set of nodes might not be obligatory at some applications – some nodes can be omitted, if it is not effective according to the creating of the routes. This route can be then formulated as a VRP with the choice of nodes. Objective function representing total cost (or total length of all routes) doesn´t solve the problem, because the optimal solution doesn´t contain any non-obligatory nodes. Neither the objective function with the total amount of transport is not eligible, because all the nodes are included in the optimal solution.

The problem was solved with a constraint that the requirements on the transport into the nodes were divided into the urgent requirements, which had to be realized immediately, and others, non-obligatory, which might not be in a solution if it caused a decrease in the efficiency of the solution (Pelikán & Jablonský, 2020; Pelikán, 2019). The goal was to minimize costs to a unit of transported items (further I_c). The model with this non-linear objective function was transformed by using the Charles-Cooper method into the linear model with binary variables. The condition for using the Charles-Cooper transformation is to have the positive denominator in the objective function for all the acceptable solutions (Barros, 1998; Martos, 1975). This can be achieved by the fact that every acceptable solution will contain at least one node (except for a depot).

There will be suggested these three models in this article:

a) a model with obligatory and optional nodes;
b) a model with given maximal limit of the efficiency index;
c) a model with given number of nodes included in the solution.

2. Mathematical Model of VRP with Obligatory and Optional Nodes

At first, we introduce the mathematical model of the problem.

Parameters of the model:
n number of nodes,
m number of optional nodes, nodes $2,3,\ldots,m$ are optional nodes, nodes $m+1,m+2,\ldots,n$ are compulsory, node 1 is depot,
d_{ij} distance between node i and node j,
q_i demand of node i,
W capacity of vehicle.

Variables of the model are:
x_{ij} binary, equals 1 if a vehicle travels from node i to node j,
u_j variables in anti-cyclic constraints.

The object function I_c (1) is ratio with denominator total amount of loads of all routes and numerator total length of all routes. Equation (2) ensures that compulsory nodes will be entered and its demand q_j is covered. Equation (3) means condition: if vehicle enters a node it has to leave it. Anti-cyclic conditions are in (4). Inequality (5) assures that capacity of vehicles is not exceeded.

$$f(x) = \frac{\sum_{i=1}^{n} \sum_{j=1}^{n} d_{ij} x_{ij}}{\sum_{i=1}^{n} \sum_{j=1}^{n} q_i \, x_{ij}} \qquad \rightarrow min \tag{1}$$

$$\sum_{i=1}^{n} x_{ij} = 1 \quad i = 1,2,\ldots,m \tag{2}$$

$$\sum_{i=1}^{n} x_{ij} = \sum_{i=1}^{n} x_{ji} \quad i = 1,2,\ldots,n \tag{3}$$

$$u_i + q_j - W\left(1 - x_{ij}\right) \le u_j \quad i = 1,2,\ldots,n, j = 2,3,\ldots,n, i \ne j \tag{4}$$

$$u_j \le W \quad j = 2,3,\ldots,n \tag{5}$$

$$x_{ij} \quad i,j = 1,2,\ldots,n, \quad i \ne j \ \ binary \tag{6}$$

Nonlinear model is solved using the Charles-Cooper method (see Pelikán and Jablonský (2020)).

The proposed mathematical model was verified on an illustrative example. Consider 11 nodes where node 1 is a depot and m=6. Capacity of each vehicle is W=100. The requirements of the nodes are q = (0, 5, 20, 10, 20, 85, 65, 30, 20, 70, 30). The distance matrix D is in Table 1.

Table 1. Distance matrix D

0	13	6	55	93	164	166	168	169	241	212
13	0	11	66	261	175	177	179	180	239	208
6	11	0	60	97	168	171	173	174	239	209
55	66	60	0	82	113	115	117	117	295	265
93	261	97	82	0	113	115	117	118	333	302
164	175	168	113	113	0	6	7	2	403	374
166	177	171	115	115	6	0	8	7	406	376
168	179	173	117	117	4	8	0	3	408	378
169	180	174	117	118	3	7	3	0	409	379
241	239	239	295	333	403	406	408	409	0	46
212	208	209	265	302	374	376	378	379	46	0

Optimal solution is:

1. route 1-3-2-4-1 with transport volume 73 and length of the route 138.
2. route 1-5-7-9-6-1 with transport volume 100 and length of the route 381.

The total length of all route is 519, total load is 173, so length on one unit of load is I_c= 3.

If we have to put optional nodes into the solution, then their choice is considerably influenced by the given obligatory nodes. In the optimal solution there will be chosen those nodes that are close to these obligatory nodes while creating the routes, where won´t be increased the total length of routes a lot by including them into the routes. On the other hand, the amount of transfer will increase.

3. Linear Model with a Restraint on the Amount of the Efficiency Index I_c

It is possible to show experimentally that by adding the optional nodes into the solution the efficiency index I_c might not only decrease, but it can increase as well. While minimizing the efficiency index I_c, there are not many or even no optional nodes in the optimal solution. In praxis it won´t be necessary to insist on the minimal value of the efficiency index and therefore it will be possible to allow a tiny increase such as the number of optional nodes would consequently increase.

Vehicle routing problem with optional nodes can be solved by the linear objective function, if there is an upper bound I_c^{max} of the total costs per volume unit I_c (costs can be represented e.g. by the length of all routes). Objective function might be e.g. the total volume of transport with its maximization. The constraint for upper bound of the efficiency index I_c

$$\sum_{i=1}^{n} \sum_{j=1}^{n} d_{ij} x_{ij} \leq I_c^{max} \sum_{i=1}^{n} \sum_{j=1}^{n} q_i x_{ij} \tag{7}$$

(which is given) will be transformed into the linear inequation in the form of (7).

4. Model with Given Number of Optional Nodes

In the previous chapter there was suggested a model, where we have given upper bound of costs to the unit of transport I_c. Therefore, it is difficult to determine the upper bound. We can go on so that we set the number of optional nodes that have to be included into the solution (their concrete choice will be solved by mathematical model). After that we can compare optimal values of the efficiency index I_c for different given numbers of optional nodes included into the solution. Mathematical model is (1)-(6) and (8).

The equation (8) assures that exactly n_0 nodes will be in the optimal solution.

$$\sum_{i=2}^{n} \sum_{j=1}^{n} x_{ij} = n_0 \tag{8}$$

Table 2. Optimal solution depending on given number of optional nodes

Solution				
n_0	C	L	I_c	Routes
1	12	20	0.60	1-3-1
2	30	25	1.2	1-3-2-1
3	344	115	2.99	1-3-1; 1-4-6-1
4	362	120	3.01	1-2-3-1; 1-6-4-1
5	375	120	3.12	1-2-3-1; 1-4-9-7-1
6	704	215	3.27	1-3-2-1; 1-6-4-1; 1-8-7-1
7	888	235	3.77	1-4-6-1; 1-5-3-2-1; 1-7-8-1
8	1203	315	3.81	1-2-3-1; 1-6-4-1; 1-7-8-1; 1-11-10-1
9	1387	335	4.14	1-2-3-5-1; 1-4-6-1; 1-7-8-1; 1-10-11-1
10	1571	355	4.425	1-5-7-1; 1-6-1; 1-8-9-4-2-3-1; 1-11-10-1

Optimal solution is shown in Table 2 where column *C* contains total length of routes, column *L* total load.

5. Heuristic Method

The VRP is NP-hard, so it is suitable to propose and use a heuristic method. One of them is nearest neighbor method (NNM), which has to be modified tor our problem. NNM is easy to implement and executes quickly, but it does not yield the optimal solution. The solution is created by gradually adding another node to the sequence of nodes obtained so far until stop rule is met.

Stop rule is: the prescribed number of nodes n_0 in the solution is reached or it is no possible to reduce the cost index I_c by adding another node.

The solution consists of one o more routes which are created gradually by adding nodes not yet included in routes. A load on each route must exceed the capacity of the vehicle.

Notation:

n' number of nodes included in some route,

s the last node of the last created route,

L' load of the last route,
N' a set of nodes yet not included in routes,
L, *C* the total load and length of yet created routes.

These are steps of NNM algorithm:
Step 1: Put n':=0, L':=0, s:=1, N':={2,3,…,n}, L:=0, C:=0.
Step 2: If $n' = n$ then stop.
Find k such that:
$$\frac{C - d_{s1} + d_{sk} + d_{k1}}{L + q_k} = \min_j \frac{C - d_{s1} + d_{sj} + d_{j1}}{L + q_j}, \text{ where } j \in N', L' + q_j \le W.$$
If k does not exists then a new route starts and put s:=1, L'=0, otherwise put
$s:=k$, $L' := L' + q_k$, $L := L + q_k$, $C := C - d_{s1} + d_{sk} + d_{k1}$, $N' := N' - \{k\}$.
Go to Step 2.

Use of NNM is shown on the Table 3.

Table 3. Solution of NNM depending on given number of optional nodes

NNM				
n_0	C	L	I_c	Routes
1	12	20	0.60	1-3-1
2	30	25	1.2	1-3-2-1
3	138	35	3.94	1-3-2-4-1
4	364	100	3.64	1-3-2-4-7-1
5	692	185	3.75	1-6-1; 1-3-2-4-7-1
6	886	205	4.32	1-5-1; 1-6-1; 1-3-2-4-7-1
7	1070	235	4.55	1-5-8-1; 1-6-1; 1-3-2-4-7-1
8	1074	255	4.2	1-5-8-9-1; 1-6-1; 1-3-2-4-7-1
9	1495	285	5.24	1-5-8-9-11-1; 1-6-1; 1-3-2-4-7-1
10	1587	355	4.47	1-10-1; 1-5-8-9-11-1; 1-6-1; 1-3-2-4-7-1

6. Conclusions

Topic of the paper is modification of vehicle routing problem in which part or all nodes are optional. It solved problem which nodes choose and include into optimal routes. A nonlinear object function I_c is minimized. Function I_c represents costs index: total costs per unit load of all routes. Two alternative approaches are proposed and illustrative example is presented.

Conflict of interest: none

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