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Comparison of Selected Models for Ranking of Decision Making Units

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Abstract: Data envelopment analysis (DEA) techniques belong to the most often applied models for ranking of decision making units (DMU) or alternatives according to their input and output characteristics. Traditional DEA models assign to the DMUs efficiency scores that allow their ranking – higher scores higher position in the final ranking. The frequently discussed problem is ranking of efficient units because they reach maximum efficiency score. Their number can be quite high depending on the number of DMUs and the number of variables (inputs and outputs) of the model. The aim of the paper is to compare the most important methods for ranking of DMUs. Their application may lead to different rankings. In this case we offer a procedure for aggregation of several different rankings into one final result. The proposed methodology will be illustrated on a numerical example and the results discussed.

Keywords: data envelopment analysis; ranking; efficiency; super-efficiency; aggregation

JEL Classification: C44

1. Introduction

DEA models is a non-parametric method for estimation of the production possibility set (PPS) frontiers, and identification of the DMUs being on the frontier (efficient DMUs) on one hand, and the remaining units (inefficient DMUs) on the other hand. First DEA models were introduced by Charnes et al. (1978), and further developed by many authors in the next decades until the present time. Charnes' et al. (1978) is known in the literature as CCR model. DEA models evaluate the set of DMUs according to their variables (inputs and outputs). Let us consider the set of *n* DMUs characterized by *m* inputs and *r* outputs. The values of the inputs for the DMUs are x_{ij}, $i = 1, \ldots, n$, $j = 1, \ldots, m$. Analogously, y_{ik}, $i = 1, \ldots, n$, $k = 1, \ldots, r$, are the output values. The traditional formulation of the DEA model in its input orientation is as follows:

Minimize:

$$\theta_q^{CCR}$$

subject to:

$$\sum_{i=1}^{n} x_{ij}\lambda_i + s_j^- = \theta_q x_{qj}, j = 1,\ldots, m, \quad (1)$$

$$\sum_{i=1}^{n} y_{ik}\lambda_i - s_k^+ = y_{qk}, \quad k = 1,\ldots, r,$$

$$\lambda_i \geq 0,\ s_j^- \geq 0,\ s_k^+ \geq 0, \quad i = 1,\dots, n, j = 1,\dots, m, k = 1,\dots, r,$$

s_j^-, $j = 1,\dots, m$, s_k^+, $k = 1,\dots, r$, are slack variables, and DMU_q is the unit under evaluation. Optimal objective function value of model (1) equals to 1 for the units belonging to the PPS frontier, and less than 1 for the inefficient units. Lower values indicate that the unit is further from the frontier. The inefficient units are easily ranked according to their efficiency scores. For the efficient ones a suitable procedure for their discrimination must be applied. There have been proposed many such procedures in the literature in the past. In our study, five procedures based on solving linear programs are considered.

Except the traditional radial DEA models – their typical representative is model (1) – have been proposed models based on measuring the efficiency using slack variables only. Tone's model (Tone, 2001) belongs to the most frequently applied in current research studies. This family of models is denoted as SBM (slacks-based measure) models. Its mathematical formulation follows:

Minimize:

$$\theta_q^{SBM} = \frac{1 - \frac{1}{m}\sum_{j=1}^{m}\left(s_j^- / x_{qj}\right)}{1 + \frac{1}{r}\sum_{k=1}^{r}\left(s_k^+ / y_{qk}\right)}.$$

subject to

$$\sum_{i=1}^{n} x_{ij}\lambda_i + s_j^- = x_{qj},\ \ j = 1,\dots, m, \tag{2}$$

$$\sum_{i=1}^{n} y_{ik}\lambda_i - s_k^+ = y_{qk},\ \ k = 1,\dots, r,$$

$$\lambda_i \geq 0,\ s_j^- \geq 0,\ s_k^+ \geq 0, \quad i = 1,\dots, n, j = 1,\dots, m, k = 1,\dots, r,$$

Model (2) returns objective function (efficiency score) equal to 1 for SBM efficient units and it is less than one for the inefficient ones. It is proved that the SBM efficiency score is always less or equal than the CCR efficiency score. Model (2) is not linear, but its linearization can easily be done using Charnes-Cooper transformation.

The paper is organized as follows. Section 2 contains formulation of all DEA models used further in numerical experiments. Section 3 presents the results of all methods on an example published in (Jablonský, 2016) that is often used for such comparisons. The final aggregated ranking is derived using an original optimization procedure. Aggregation of rankings is a frequently discussed problem in current research. One of the last papers in top OR journals about this topic is (Mohammadi & Rezaei, 2020). The paper concludes by discussion of results and future research possibilities.

2. Methodology

Standard DEA models as model (1) and (2) allow (CCR or SBM) ranking of inefficient units according to their efficiency scores. To rank efficient units, because to their maximum efficiency scores, many models based on various principles have been proposed in the past. For comparison purposes we will apply the following ones:

1. Andersen and Petersen's (1993) super-efficiency model – AP model. This model removes the unit under evaluation from the set of DMUs and measures the distance of this unit from the new PPS frontier. Higher distance means that higher increasing inputs or decreasing outputs does not affect the efficient status of the unit, i.e. the unit under evaluation has higher super-efficiency score. The input-oriented formulation of the AP model is the same as model (1). Only difference is in putting the weight of the unit under evaluation equal to zero. As the result, the super-efficiency score of the originally efficient unit is greater than 1.
2. Tone's (2002) super-efficiency model – SSBMT model. It is, as in the previous case, a model from the category of super-efficiency models. The objective function of this model equals to 1 if the unit under evaluation is SBM inefficient, i.e. has SBM efficiency score computed by model (2) less than 1. For SBM efficient units returns a super-efficiency score greater than 1 which allows complete ranking of all DMUs. The non-linear formulation of this model is below:

   Minimize:

$$\phi_q^{SBM} = \frac{\frac{1}{m}\sum_{j=1}^{m} x_j^* / x_{qj}}{\frac{1}{r}\sum_{k=1}^{r} y_k^* / y_{qk}}$$

   subject to:

$$\sum_{i=1, i\neq q}^{n} x_{ij}\lambda_i + s_j^- = x_j^*,\ \ j = 1,\dots, m, \tag{3}$$

$$\sum_{i=1, i\neq q}^{n} y_{ik}\lambda_i - s_k^+ = y_k^*,\ \ k = 1,\dots, r,$$

$$x_{qj} \le x_j^*,\ \ j = 1,\dots, m,$$

$$y_{qk} \le y_k^*,\ \ k = 1,\dots, r,$$

$$\lambda_i \ge 0,\ \ s_j^- \ge 0,\ s_k^+ \ge 0,\ \ i = 1,\dots, n, j = 1,\dots, m, k = 1,\dots, r,$$

   As the SBM model (2), model (3) can be transformed into a linear program easily. Moreover, its input- and output-orientation versions have been proposed in (Tone, 2002).

3. Jablonský (2012) formulated a super-efficiency goal programming model (SBMG model) that can be used for comparison purposes with other ranking models. Its mathematical formulation follows:

   Minimize:

$$\varphi_q^G = 1 + tD + (1-t)\left(\sum_{j=1}^{m}\left[s_{j1}^{+} / x_{qj}\right] + \sum_{k=1}^{r}\left[s_{k2}^{-} / y_{qk}\right]\right),$$

   subject to:

$$\sum_{i=1, i \neq q}^{n} x_{ij}\lambda_i + s_{j1}^{-} - s_{j1}^{+} = x_{qj}, \;\; j = 1,\ldots, m, \tag{4}$$

$$\sum_{i=1, i \neq q}^{n} y_{ik}\lambda_i + s_{k2}^{-} - s_{k2}^{+} = y_{qk}, \;\; k = 1,\ldots, r,$$

$$s_{j1}^{+} \leq Dx_{qj}, s_{k2}^{-} \leq Dy_{qk} \;\; j = 1,\ldots, m, k = 1,\ldots, r,$$

$$s_{j1}^{-}, s_{j1}^{+}, s_{k2}^{-}, s_{k2}^{+} \geq 0, \lambda_i \geq 0, j = 1,\ldots, m, k = 1,\ldots, r, í = 1,\ldots, n,$$

$$t \in \{0, 1\}.)$$

   where $s_{j1}^{-}, s_{j1}^{+}, s_{k2}^{-}, s_{k2}^{+}$ are variables measuring the negative and positive deviations of the virtual unit and the unit under evaluation in input and output space. *D* is the maximum relative deviation, and *t* is the parameter that may be set to 0 or 1. The value $t = 0$ ensures minimization of the sum of relative deviations, and $t = 1$ minimizes the maximum deviation. The model is applied on CCR (or SBM) efficient units and returns super/efficiency score greater or equal than 1.

4. An interesting concept that allows complete ranking of the DMUs is measuring the distance of the units from the pessimistic frontier introduced by Wang et. al (2007). Pessimistic frontier is the opposite of the optimistic PPS frontier constructed by CCR model (1).

   Maximize:

$$\theta_q^P$$

   subject to:

$$\sum_{i=1}^{n} x_{ij}\lambda_i \geq \theta_q x_{qj}, \;\; j = 1,\ldots, m, \tag{5}$$

$$\sum_{i=1}^{n} y_{ik}\lambda_i \leq y_{qk}, \;\; k = 1,\ldots, r,$$

$$\lambda_i \geq 0,\ s_j^- \geq 0,\ s_k^+ \geq 0,\ i = 1, \ldots, n, j = 1, \ldots, m, k = 1, \ldots, r.$$

The optimal value θ_q^P is greater than 1. Higher values indicate that the unit under evaluation is further from the frontier. Hence higher values of the pessimistic model lead to higher ranking of the DMUs. It is possible to formulate a pessimistic super-efficiency model in a similar way as in the optimistic case. This model allows distinguishing among pessimistic efficient units.

Cross efficiency evaluation is an approach based on completely different principles than super-efficiency models. In this approach the unit under evaluation is evaluated using the optimal weights of the other unit of the set. Let E_{qj} be the efficiency score of the q-th unit using the optimal weights of the j-th unit derived using traditional CCR model. The final result of the evaluation for the q-th unit is the average cross-efficiency score computed as a simple average

$$\varphi_q = \frac{\sum_{j=1}^{n} E_{qj}}{n},\ q = 1, \ldots, n. \tag{6}$$

The maximum value of φ_q scores is 1. Higher values show higher level of efficiency and higher ranking.

3. Results

In this section, the results of all five algorithms are computed with the dataset that contains 19 DMUs and 5 variables (2 inputs and 3 outputs). This dataset originates from the paper (Jablonský, 2016) and due to the limited space is not displayed here. Because the results of all methods are different, the final ranking will be derived using an optimization procedure that minimizes the sum of deviations of the final ranking and all five rankings obtained by the methods mentioned in the previous section.

Table 1 contains efficiency and super-efficiency scores. In the first column, there are CCR efficiency scores (less than 1) computed by model (1), and for CCR efficient units (units 1, 11, and 18) the super-efficiency scores derived by Andersen and Petersen (1993) model (greater than 1). Second column of Table 2 contains the efficiency scores obtained by SBM model (2), and for the SBM (and CCR efficient also) efficient units their super/efficiency scores derived by model (3). The SBMG model (4) is just the model that may be used for CCR efficient units to distinguish among them. That is why, the results in the third column of Table 3 are chosen as a geometric average of CCR and SBM efficiency scores, except for the CCR efficient units. The fourth column presents the scores given by the pessimistic model (5) and by its super-efficiency modification. Higher values in this column indicate that the unit is further from the pessimistic frontier, and it is better evaluated in the final ranking. The last column contains cross-efficiency scores computed by (6).

Table 1. Efficiency and super-efficiency scores

DMUs	CCR/AP	SBM/SSBM	SBMG	PESSIM	CROSS
1	1.389	1.103	1.305	1.031	0.698
2	0.618	0.464	0.535	0.807	0.520
3	0.985	0.850	0.915	1.588	0.858
4	0.977	0.828	0.899	1.293	0.766
5	0.837	0.688	0.759	1.395	0.761
6	0.875	0.620	0.737	1.401	0.754
7	0.657	0.415	0.522	0.979	0.531
8	0.773	0.688	0.729	1.278	0.708
9	0.972	0.777	0.869	1.583	0.857
10	0.853	0.672	0.757	1.254	0.727
11	1.407	1.225	1.521	1.580	0.984
12	0.788	0.642	0.711	1.216	0.701
13	0.674	0.390	0.513	0.882	0.533
14	0.693	0.524	0.603	1.103	0.619
15	0.850	0.721	0.783	1.119	0.685
16	0.932	0.828	0.878	1.393	0.804
17	0.948	0.689	0.808	1.370	0.795
18	1.319	1.168	1.386	1.289	0.926
19	0.833	0.723	0.776	1.190	0.711

Table 2 contains ranking of all units according to the scores in Table 1. Their analysis shows quite high differences in the positions of the DMUs. That is why a simple optimization procedure for aggregation of rankings is proposed. The model that results in final ranking of n units that minimizes the sum of all positive and negative deviations from m particular rankings is formulated as follows:

Minimize:

$$\sum_{i=1}^{m}\sum_{j=1}^{n}(d_{ij}^{-}+d_{ij}^{+})$$

subject to:

$$\begin{aligned}
&x_{ij}+d_{ij}^{-}-d_{ij}^{+}=y_i, && i=1,\dots,m, j=1,\dots,n,\\
&\sum_{j=1}^{n}z_{ij}=1, && i=1,\dots,m,\\
&\sum_{i=1}^{m}z_{ij}=1, && j=1,\dots,n,\\
&y_i=\sum_{j=1}^{n}j.z_{ij}, && i=1,\dots,m,\\
&z_{ij}-binary,
\end{aligned} \tag{7}$$

where x_{ij}, $i = 1,\dots, n$, $j = 1,\dots, m$, is the position of the i-th unit in the j-th ranking, y_i, i = 1,…, m, is the final position of the i-th unit, and d_{ij}^{-}, d_{ij}^{+} are deviational variables to be minimized. z_{ij}, $i = 1,\dots, n$, $j = 1,\dots, m$, are artificial binary variables that allows to ensure the uniqueness of the final ranking. Model (7) can be easily modified to find the solution that minimizes the absolute value of the maximum deviation. Both aggregated rankings (SUM and MAXMIN) are presented in the last two columns of Table 2.

Table 2. Ranking of the DMUs

DMUs	CCR/AP	SBM/SSBM	SBMG	PESSIM	CROSS	SUM	MINMAX
1	2	3	3	16	14	3	9
2	19	17	17	19	19	19	19
3	4	4	4	1	3	4	3
4	5	6	5	8	7	7	6
5	12	11	11	5	8	10	11
6	9	15	13	4	9	9	8
7	18	18	18	17	18	18	18
8	15	12	14	10	12	14	14
9	6	7	7	2	4	5	4
10	10	13	12	11	10	12	10
11	1	1	1	3	1	1	1
12	14	14	15	12	13	15	15
13	17	19	19	18	17	17	17
14	16	16	16	15	16	16	16
15	11	9	9	14	15	13	12
16	8	5	6	6	5	6	5
17	7	10	8	7	6	8	7
18	3	2	2	9	2	2	2
19	13	8	10	13	11	11	13

4. Discussion and Conclusions

The results presented in Table 2 show the expected outcome – the similarity in rankings among the ones obtained by the family of super-efficiency models is very high. On the contrary, cross-efficiency approach produces in some cases significant differences. It is not so surprising because this model is based on completely different principles than the super-efficiency models. Moreover, the results given by this model need not be always unique because of not rarely occurring alternative solutions the differences. Pessimistic frontier model is closer to the CCR results but there is an exception. In our case, it is the first unit of the set. This unit is efficient in the optimistic (CCR) model and pessimistic model as well. It means that this unit is the member of both (optimistic and pessimistic) frontiers. This is the reason why this unit is ranked much worse in the pessimistic model than in the optimistic one.

The model for the aggregation of several different rankings is a tool that can allow to get one final ranking, e.g. if several decision makers express their opinions and try to find a compromise solution. In the case of our example, both aggregated rankings are similar each other except for the first unit where the difference is six positions. It is caused by worse positions of this unit in pessimistic and cross-evaluation models.

The models for aggregation of rankings are both easy to solve even they are discrete optimization models. The numerical experiments confirm that they produce the final ranking for the problems with up to 200 units and up to 10 single rankings in few second using simple discrete solvers. Ranking of the DMUs in DEA models is still a frequently discussed problem. The future research may be concentrated on the ranking of the units in network production systems.

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Conflict of interest: none

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