mathematics

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On Terms of Generalized Fibonacci Sequences which are Powers of their Indexes

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Received: 29 June 2019; Accepted: 31 July 2019; Published: 3 August 2019

Abstract: The k-generalized Fibonacci sequence $(F_n^{(k)})_n$ (sometimes also called k-bonacci or k-step Fibonacci sequence), with $k \geq 2$, is defined by the values $0, 0, \ldots, 0, 1$ of starting k its terms and such way that each term afterwards is the sum of the k preceding terms. This paper is devoted to the proof of the fact that the Diophantine equation $F_m^{(k)} = m^t$, with $t > 1$ and $m > k + 1$, has only solutions $F_{12}^{(2)} = 12^2$ and $F_9^{(3)} = 9^2$.

Keywords: k-generalized Fibonacci sequence; Diophantine equation; linear form in logarithms; continued fraction

MSC: Primary 11J86; Secondary 11B39.

1. Introduction

The well-known Fibonacci sequence $(F_n)_{n \geq 0}$ is given by the following recurrence of the second order

$$F_{n+2} = F_{n+1} + F_n,$$

for $n \geq 0$, with the initial terms $F_0 = 0$ and $F_1 = 1$. Fibonacci numbers have a lot of very interesting properties (see e.g., book of Koshy [1]).

One of the famous classical problems, which has attracted a attention of many mathematicians during the last thirty years of the twenty century, was the problem of finding perfect powers in the sequence of Fibonacci numbers. Finally in 2006 Bugeaud et al. [2] (Theorem 1), confirmed these expectations, as they showed that $0, 1, 8$ and 144 are the only perfect powers in the sequence of Fibonacci numbers. This result is usually given in the relevant mathematical literature as the Fibonacci Perfect Powers Theorem. The result itself is extremely interesting, but the way of its proof is even more interesting to mathematicians, as this proof combined two powerful techniques from number theory, namely, Baker's theory on linear forms in logarithms and the tools from the Wiles's proof of the Last Fermat Theorem.

This result started great efforts for finding perfect powers in some generalized Fibonacci sequences. Luca and Shorey [3], Theorem 2 showed that products of two or more consecutive terms in the Fibonacci sequence is a perfect power only for the trivial case $F_1 F_2 = 1$. Marques and Togbé [4] found that Fibonomial coefficients $\{{m \atop k}\}$, defined by

$$\left\{ {m \atop k} \right\} = \frac{F_m \cdots F_{m-k+1}}{F_1 \cdots F_k},$$

where m and k are non-negative integers, $0 \leq k \leq m$, and $\{{m \atop 0}\} = 1$, are not a perfect power for $m > k + 1 > 2$. Marques and Togbé [5] found all Fibonacci and Lucas numbers written in the form $2^a + 3^b + 5^c$ and Qu, Zeng, and Cao [6] generalized this result for the form $2^a + 3^b + 5^c + 7^d$.

Let $F^{(k)}$, $k \geq 2$, denote a sequence $(F_n^{(k)})_{n \geq -(k-2)}$ of the k-generalized Fibonacci numbers, whose terms satisfy the following recurrence relation

$$F_{n+k}^{(k)} = F_{n+k-1}^{(k)} + F_{n+k-2}^{(k)} + \cdots + F_n^{(k)},$$

with values $0, 0, \ldots, 0, 1$ of starting k its terms and with the first nonzero term $F_1^{(k)} = 1$.

Recently, a lot of works were devoted to the sequences $F^{(k)}$. In particular, in 2013, Bravo and Luca [7] and Marques [8] confirmed (independently) a conjecture, proposed by Noe and Post [9], on coincidences between terms of these sequences.

Chaves and Marques [10] proved that the Diophantine equation $(F_n^{(k)})^2 + (F_{n+1}^{(k)})^2 = F_m^{(k)}$ has no solution in positive integers m, n, k for $n > 1$ and $k \geq 3$. This result was generalized by Bednařík et al. [11], they showed that the Diophantine equation $(F_n^{(k)})^2 + (F_{n+1}^{(k)})^2 = F_m^{(l)}$ has no solution in positive integers m, n, k, l for $n > 1$ and $2 \leq k < l$. Chaves and Marques [12] generalized [10] in a different way, as authors dealt with the Diophantine equation $(F_n^{(k)})^s + (F_{n+1}^{(k)})^s = F_m^{(k)}$ in positive integers m, n, k, s and they proved that it has no solution if $3 \leq k \leq \max(n, \log s)$ (this equation was solved completely in [13]).

Despite all the above mentioned achievements, finding a complete solution to the following Diophantine equation remains an open problem

$$F_m^{(k)} = y^t. \tag{1}$$

The most important reason for that is probably in the fact that the method of Bugeaud et al. [2] cannot be applied to $F^{(k)}$ for $k \geq 3$, as the proof of the case $k = 2$ is related to ternary Diophantine equation with signature $(p, p, 2)$ and it use a lot of identities and divisibility properties for $F^{(2)}$, which we do not have for $k \geq 3$.

In the last decade, some authors have studied special cases of (1). Bravo and Luca [14] created a method to solve the equation $F_m^{(k)} = 2^t$ and Marques and Trojovský [15] solved the case $F_m^{(k)} = k^t$.

In this paper we continue in this project, as we work on Equation (1) for $y = m$. More precisely, our main result is the following

Theorem 1. *Let m, k, t be any integers, with $m > k + 1$. Then the only solutions of the Diophantine equation*

$$F_m^{(k)} = m^t \tag{2}$$

are

$$(m, k, t) \in \{(12, 2, 2), (9, 3, 2)\}.$$

The condition $m > k + 1$ is established in Theorem 1 to avoid the uninteresting solutions related to m being a power of two. For instance, if p is any odd prime number, then $(m, k, t) = (2^p, k, (2^p - 2)/p)$ is a solution of Equation (2) for all $k \geq 2^p - 1$.

Now we describe our proof of Theorem 1. Using Dresden and Du [16] (Formula (2)), we obtain an upper bound for a linear form in three logarithms related to Equation (1). Then, using a lower bound due to Matveev we gain an upper bound of t in terms of m and k. Next, we use a similar method as in [14], but in our proof we get an upper bound for a linear form in two logarithms and our case is more complicated, as we need need to find an upper bound for $|2^{n-2} - m^t|$. Then, by a result due to Laurent we find an absolute upper bound for k in terms of $\log m$ and consequently to gain an absolute upper bound for m, k and t. Finally, with the help of some facts on convergent of continued fractions we can improve the upper bound for k in terms of t and a constant. The computations in the paper were performed using Mathematica ® (see [17]).

We remark that the main difference between this work and the paper in [15] is that the case $y = k$ in (1) is easier, since the growth of $F_n^{(k)}$ (which is 2^{n-2}) does not depend on k. Also, the nature of the polynomial-exponential equation in the case $y = k$ is computationally better in order to use a reduction method (to find all solutions), since the upper bound for k is substantially smaller than the one for m.

Our main approach of the proof of Theorem 1 is a similar as in [14], as we think that this kind of approach is very helpful to the readers.

2. Auxiliary Results

First of all, we shall recall some tools and facts which we use hereafter.

It is well-known that the characteristic polynomial

$$\psi_k(x) := x^k - x^{k-1} - \cdots - x - 1$$

of $(F_n^{(k)})_n$ is irreducible over $\mathbb{Q}[x]$ (see [18]) and it has just one single and dominant zero α outside the unit circle. This zero α is located between $2(1 - 2^{-k})$ and 2 (see [19]). Also, Dresden and Du [16] (Theorem 1) gave a simplified "Binet-like" formula for $F_n^{(k)}$ in the following form

$$F_n^{(k)} = \sum_{i=1}^{k} \frac{\alpha_i - 1}{2 + (k+1)(\alpha_i - 2)} \alpha_i^{n-1}, \tag{3}$$

where $\alpha_1 = \alpha, \alpha_2, \ldots, \alpha_k$ are the roots of $\psi_k(x)$. Further, in [20] (Lemma 1) it was proved that

$$\alpha^{n-2} \leq F_n^{(k)} \leq \alpha^{n-1}, \text{ for all } n \geq 1. \tag{4}$$

Clearly, the roots of $\psi_k(x)$ inside the unit circle have almost negligible contribution in formula (3). More precisely, Dresden and Du [16] proved that

$$|F_n^{(k)} - g(\alpha, k)\alpha^{n-1}| < \frac{1}{2},$$

where we adopt the notation $g(x, y) := (x - 1)/(2 + (y + 1)(x - 2))$.

The main powerful tool to prove Theorem 1 is a lower bound for a linear form logarithms à la Baker, which was given by the following result of Matveev (see [21] or [2] (Theorem 9.4)).

In the following, we shall use the a more accurate lower bound for linear forms in three logarithms to prove our main result

Lemma 1. *Let $\gamma_1, \gamma_2, \gamma_3$ be real algebraic numbers and let b_1, b_2, b_3 be nonzero integer numbers. Define*

$$\Lambda = b_1 \log \gamma_1 + b_2 \log \gamma_2 + b_3 \log \gamma_3.$$

Let D be the degree of the number field $\mathbb{Q}(\gamma_1, \gamma_2, \gamma_3)$ over $\mathbb{Q}$ and let A_1, A_2, A_3 be any positive real numbers, which satisfy the following conditions

$$A_j \geq \max\{Dh(\gamma_j), |\log \gamma_j|, 0.16\}, \text{for } j = 1, 2, 3.$$

Assume that

$$B \geq \max\{1, \max\{|b_j|A_j/A_1; 1 \leq j \leq 3\}\}.$$

Define also

$$C_1 = 6750000 \cdot e^4(20.2 + \log(3^{5.5}D^2 \log(eD))).$$

If $\Lambda \neq 0$, *then*

$$\log|\Lambda| \geq -C_1 D^2 A_1 A_2 A_3 \log(1.5eDB \log(eD)).$$

We used in the previous lemma, the logarithmic height of an n-degree algebraic number γ, which is defined as

$$h(\gamma) = \frac{1}{n}(\log|a_0| + \sum_{j=1}^{n} \log\max\{1, |\gamma^{(j)}|\}),$$

where a_0 is the leading coefficient of the minimal polynomial of γ (over $\mathbb{Z}$) and $(\gamma^{(j)})_{1\leq j\leq n}$ are the conjugates of γ.

Finally, we summarize all necessary notations and previous results and we can start the proof of our Theorem 1.

3. Some Key Lemmas

3.1. Upper Bounds for m in Terms of t and for t in Terms of k and m

The aim of this subsection is to prove the following result

Lemma 2. *If* (m, k, t) *is an integer solution satisfying Diophantine Equation* (2), *with* $m > k + 1$, *then*

$$m < (5t + 4)\log(2.5t + 2) \text{ and } t < 8.6 \cdot 10^{11} k^4 \log^2 k \log m.$$

Proof. Of course, we can consider $k \geq 3$. By using (4), we get $\alpha^{m-2} \leq m^t$ and after applying the *log* function

$$\frac{m}{\log m} < 2.5t + 2,$$

where we used that $\alpha > 7/4$. Now we can use the following result on function $x/\log x$ (from [14], p. 74). Since the function $x/\log x$ is increasing for $x > e$ the following holds

$$\frac{x}{\log x} < A \text{ implies that } x < 2A\log A. \quad (5)$$

Thus, setting $x := m$ and $A := 2.5t + 2$ in (5), we get $m < (5t + 4)\log(2.5t + 2)$.

Now, by using Equation (2) together with Equation (3) we obtain

$$g(\alpha, k)\alpha^{m-1} - m^t = E_k(m) \in (-1/2, 1/2),$$

where $E_k(m) := \sum_{i=2}^{k} g(\alpha_i, k)\alpha_i^{m-1}$. Thus

$$\left|\frac{g(\alpha, k)\alpha^{m-1}}{m^t} - 1\right| < \frac{1}{m^t},$$

where we used that $|E_k(m)| < 1/2$. Thus

$$|e^{\Lambda} - 1| < \frac{1}{m^t}, \quad (6)$$

where $\Lambda := \log(g(\alpha, k)) - t \log m + (m-1)\log\alpha$.

Now, we would like to use Lemma 1. So, we take

$$\gamma_1 := g(\alpha, k),\ \gamma_2 := m,\ \gamma_3 := \alpha$$

and

$$b_1 := 1,\ b_2 := -t,\ b_3 := m-1.$$

Thus, we have $D = [\mathbb{Q}(\alpha) : \mathbb{Q}] = k$, $h(\gamma_2) = \log m$ and $h(\gamma_3) < 0.7/k$. To find an estimate for $h(\gamma_1)$, we use the following estimate for $h(g(\alpha, k))$, which was given in ([14], p. 73)

$$h(\gamma_1) = h(g(\alpha, k)) < \log(4k + 4).$$

Hence, we can take $A_1 := k \log(4k + 4)$, $A_2 := k \log m$ and $A_3 := 0.7$.

Note that

$$\max\{1, \max\{|b_j|A_j/A_1; 1 \le j \le 3\}\} = \frac{t \log m}{\log(4k+4)} \le \frac{t \log m}{\log 17} =: B.$$

To finally apply Lemma 1, we have to prove that $g(\alpha, k)\alpha^{m-1}/m^t \ne 1$. Suppose, towards a contradiction, that $g(\alpha, k)\alpha^{m-1} = m^t$. Then, we can conjugate $g(\alpha, k)\alpha^{m-1} = m^t$ in $\mathbb{Q}(\alpha)$ to obtain

$$5^t \le m^t = |g(\alpha_i, k)||\alpha_i|^{m-1} < 1,$$

where $i > 1$, leading to an absurd since $t > 1$. Now, we can really use Lemma 1. The straightforward calculation leads to

$$\log|\Lambda| > -1.75 \cdot 10^{10} k^4 \log^2 k \log(1.44kt \log m \log(ek)), \quad (7)$$

where we used that $\log(4k + 4) < 2.6 \log k$ (we still have $k \ge 3$).

By combining (6) and (7), we obtain

$$t \log m < 1.75 \cdot 10^{10} k^4 \log^2 k \log(1.44kt \log m \log(ek))$$

and after clear simplification the rest of assertion follows. □

3.2. The Small Cases $2 \le k \le 343$

Lemma 3. *If (m, k, t) is an integer solution satisfying Diophantine Equation (2), with $m > k + 1$ and $2 \le k \le 343$, then*

$$(m, k, t) \in \{(12, 2, 2), (9, 3, 2)\}.$$

Proof. If $k \le 343$, then by Lemma 2 we get $t < 4.06 \cdot 10^{23} \log m$ and so

$$m < (5 \cdot 4.06 \cdot 10^{23} \log m + 4) \log(2.5 \cdot 4.06 \cdot 10^{23} \log m + 2)$$

leading to $m < 7.75 \cdot 10^{27}$ and consequently to $t < 2.61 \cdot 10^{25}$. □

3.3. An Upper Bound for k in Terms of $\log m$ and t in Terms of m

By hypothesis, we have that $k < m - 1$, however, in this subsection, we shall show the following stronger result (we remark that $m - 1 > 108.6 \log^3 m$ for all $m \geq 196688$):

Lemma 4. *If (m, k, t) is an integer solution satisfying Diophantine Equation (2), with $t > 1$ and $m > k + 1$. Then $t = \lfloor (m-1) \log 2 / \log m \rfloor$ and*

$$k < 108.6 \log^3 m.$$

Proof. Since $m^t > (7/4)^{m-2}$, then $t > (m-2)\log(7/4)/\log m > 0.4\sqrt{m}$, for $m \geq 5$. Combining this inequality with Lemma 2, we have

$$\frac{m}{\log^2 m} < 4.7 \cdot 10^{24} k^8 \log^4 k.$$

Since the function $x/\log^2 x$ is increasing for $x > e$, it is easy to prove that

$$\frac{x}{\log^2 x} < A \implies x < 12.5 A \log^2 A. \tag{8}$$

In fact, on the contrary, i.e., if $x \geq 12.5 A \log^2 A$, then

$$\frac{x}{\log^2 x} \geq \frac{12.5 A \log^2 A}{\log^2(12.5 A \log^2 A)} > \frac{12.5 A \log^2 A}{12.5 \log^2 A} = A,$$

which contradicts our inequality. Here we used that $\log(12.5 A \log^2 A) < \sqrt{12.5} \log A$, since $\log(12.5) + 2 \log\log A < (\sqrt{12.5} - 1) \log A$, for $A > e$.

Thus, by using (8) for $x := m$ and $A := 4.7 \cdot 10^{24} k^8 \log^4 k$, we have that

$$m < 12.5(4.7 \cdot 10^{24} k^8 \log^4 k) \log^2(4.7 \cdot 10^{24} k^8 \log^4 k).$$

Now, the inequality $\log^2(4.7 \cdot 10^{24} k^8 \log^4 k) < 61 \log k$, for $k \geq 5$, yields

$$m < 3.6 \cdot 10^{27} k^8 \log^5 k.$$

By Lemma 3, from now on we may consider $k \geq 344$ and then we have

$$m < 3.6 \cdot 10^{27} k^8 \log^5 k < 2^{k/2}.$$

By the key argument from ([14], p. 77–78), we have

$$|2^{m-2} - m^t| < \frac{5 \cdot 2^{m-2}}{2^{k/2}},$$

where we needed greatly the condition $m > k + 1$.

After dividing by 2^{m-2}, we get

$$|1 - m^t \cdot 2^{-(m-2)}| < \frac{5}{2^{k/2}}.$$

If $m^t/2^{m-2} \in (-\infty, 1/2] \cup [2, +\infty)$, then $|1 - m^t \cdot 2^{-(m-2)}| \geq 1$ yielding $2^{k/2} < 5$ which is a contradiction, since $k \geq 344$. Thus, we have $2^{m-3} < m^t < 2^{m-1}$. Now, by applying the log function and using that $(m-1)\log 2/\log m - (m-3)\log 2/\log m = 2\log 2/\log m < 1$, for $m \geq 5$, we get that there is a unique possible value for t, namely

$$t = \left\lfloor \frac{(m-1)\log 2}{\log m} \right\rfloor.$$

From now on, t will denote the value above.

Now, we rewrite the above inequality as

$$|e^{\Lambda^*} - 1| < \frac{5}{2^{k/2}},$$

where $\Lambda^* = t\log m - (m-2)\log 2$. Since for $x < 0$, it holds that $|1 - e^x| = 1 - e^{-|x|}$. To avoid unnecessary repetition, we may suppose that $\Lambda^* > 0$. Then $\Lambda^* < e^{\Lambda^*} - 1 < 5/2^{k/2}$ and by applying the natural logarithm function, we arrive to

$$\log|\Lambda^*| < \log 5 - \frac{k}{2}\log 2. \tag{9}$$

Now, we find a lower bound for $\log|\Lambda^*|$. We can use a result due to Laurent [22] (Corollary 2) for $m = 24$ and $C_2 = 18.8$ (assumptions are met because 5 and 2 are multiplicatively independent). In the first instance We have to introduce some notations. Let α_1, α_2 be real algebraic numbers, satisfying $|\alpha_j| \geq 1$, b_1, b_2 be positive integers and

$$\Gamma = b_2 \log\alpha_2 - b_1\log\alpha_1.$$

Let A_1 and A_2 be real numbers, which satisfy

$$\log A_j \geq \max\{h(\alpha_j), |\log\alpha_j|/D, 1/D\},$$

where $j \in \{1, 2\}$ and D is the degree of the number field $\mathbb{Q}(\alpha_1, \alpha_2)$ over $\mathbb{Q}$. Let us further define

$$b' = \frac{b_1}{D\log A_2} + \frac{b_2}{D\log A_1}.$$

Laurent's result asserts that if α_1, α_2 are multiplicatively independent, then

$$\log|\Gamma| \geq -18.8 \cdot D^4 \left(\max\{\log b' + 0.38, m/D, 1\}\right)^2 \cdot \log A_1 \log A_2.$$

Then we set

$$b_1 = m-2,\ b_2 = t,\ \alpha_1 = 2,\ \alpha_2 = m.$$

Hence, $D = 1$ and we can take $\log A_1 = \log 2$ and $\log A_2 = \log m$. Then we get

$$b' = \frac{m-2}{\log m} + \frac{t}{\log 2} < 2.1(m-2),$$

where we used that $m \geq 5$ and $t \leq m-2$.

Due to [14]), we may suppose that m is not a power of 2. Thus, m and 2 are multiplicatively independent and by ([22], Corollary 2), we have

$$\log|\Lambda^*| \geq -13.1 \cdot (\max\{\log(2.2(m-1)) + 0.38, 24\})^2 \log m. \tag{10}$$

Combining estimates (9) and (10) to gain

$$k \leq 108.6 \log^3 m.$$

□

4. The Proof of Theorem 1

By combining Lemma 2 with Lemma 4, we obtain the following absolute upper bounds

$$m < 10^{53},\ k < 2 \cdot 10^8, \text{ and } t < 5.9 \cdot 10^{49}. \tag{11}$$

Now, we use (9) to get

$$\left| \frac{\log m}{\log 2} - \frac{m-2}{t} \right| < \frac{5}{2^{k/2} t \log 2}.$$

We claim that there is no solution of Equation (2), for $k \geq 344$, if $(2^{k/2} \log 2)/5 > (A+2)t$, with

$$A := \max_{\substack{1 \leq s \leq 240 \\ 5 \leq m \leq 10^{53}}} \{a_{s+1,m}\},$$

where $[a_{1,m}; a_{2,m}, \ldots]$ denotes the continued fraction expansion of $\log m / \log 2$.

In fact, on the contrary, we would have

$$\left| \frac{\log m}{\log 2} - \frac{m-2}{t} \right| < \frac{1}{(A+2)t^2}. \tag{12}$$

By Legendre's criterion for continued fractions, the previous inequality implies that $(m-2)/t$ is a convergent of the continued fraction of $\log m / \log 2$, i.e., $(m-2)/t = p_{\ell,m}/q_{\ell,m}$ for some $\ell > 0$ (here $p_{s,m}/q_{s,m}$ denotes the s-th convergent of the continued fraction of $\log m / \log 2$). Since $\gcd(p_{\ell,m}, q_{\ell,m}) = 1$, then $q_{\ell,m} \mid t$ and therefore we used the upper bound for t from (11) to obtain

$$\left(\frac{1+\sqrt{5}}{2} \right)^{\ell-2} \leq F_\ell \leq q_{\ell,m} \leq t < 5.9 \cdot 10^{49}.$$

The previous inequality yields $\ell \leq 240$. On the other hand, the following well-known fact on continued fraction

$$\left| \frac{\log m}{\log 2} - \frac{p_{\ell,m}}{q_{\ell,m}} \right| > \frac{1}{(a_{\ell+1,m}+2) q_{\ell,m}^2}$$

leads to

$$\left| \frac{\log m}{\log 2} - \frac{m-2}{t} \right| > \frac{1}{(A+2)t^2}, \tag{13}$$

where we used that $q_{\ell,m} \leq t$ and $A \geq a_{\ell+1,m}$, since $\ell \leq 240$. However, inequalities (12) and (13) lead to an absurdity.

So, we obtain $(2^{k/2} \log 2)/5 \leq (A+2)t$, or equivalently, $k \leq 2.9 \log(7.3(A+2)t)$. By using computational tools we can see that $A < 5.3 \cdot 10^{74}$ and then

$$k < 2.9 \log(7.3(5.3 \cdot 10^{74} + 2) \cdot 5.9 \cdot 10^{59}) < 903.856$$

yielding $k \leq 903$. Now, we iterate the previous lemmas to obtain

$$m < 5.78 \cdot 10^{29} \text{ and } t < 1.82 \cdot 10^{27}.$$

Now, we proceed exactly as before to obtain that $k \leq 2.9 \log(7.3(A' + 2)t)$, where

$$A' := \max_{\substack{1 \leq s \leq 132 \\ 5 \leq m \leq 5.78 \cdot 10^{29}}} \{a_{s+1,m}\}.$$

Again, $A' < 7.4 \cdot 10^{49}$ and so $k \leq 520$. We repeat this argument until arriving at $k \leq 343$, which is a contradiction (by Lemma 3). Therefore, the proof is complete. □

5. Conclusions

In this paper we have been interested in finding powers of their indexes, which appear in k-generalized Fibonacci sequences. Thus, we have studied the Diophantine equation $F_m^{(k)} = m^t$ in positive integers k, m, t, with $k \geq 2$, $t > 1$, and $m > k + 1$. We have showed that this Diophantine equation has only two solutions $F_{12}^{(2)} = 12^2$ and $F_9^{(3)} = 9^2$. Our proof has been based on a linear form in logarithms, a result due to Laurent and some facts on convergents of continued fractions.

Funding: The author was supported by the Project of Excelence PrF UHK, University of Hradec Králové, Czech Republic 01/2019.

Acknowledgments: The author thanks anonymous referees.

Conflicts of Interest: The author declares no conflict of interest.

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