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Research Article

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On Diophantine equations involving Lucas sequences

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Abstract: In this paper, we shall study the Diophantine equation $u_n = R(m)P(m)^{Q(m)}$, where u_n is a Lucas sequence and R, P and Q are polynomials (under weak assumptions).

Keywords: p-adic order, Fibonacci number, Lucas sequence, order of appearance, tridiagonal matrix

MSC: 11B37, 11B39

1 Introduction

Fix two relatively prime integers a and b and let $(u_n)_{n\geq 0}$ be the Lucas sequence with characteristic polynomial $f(x) = x^2 - ax - b$, i.e., $(u_n)_{n\geq 0}$ is the integral sequence satisfying $u_0 = 0$, $u_1 = 1$, and $u_n = au_{n-1} + bu_{n-2}$, for all integers $n \geq 2$. Suppose that this sequence is non degenerated, i.e., $a \neq 0$ and α/β is not a root of unity, where α and β are the roots of $f(x)$. The most famous example of a Lucas sequence is the Fibonacci sequence $(F_n)_{n\geq 0}$ given by the recurrence relation $F_{n+2} = F_{n+1} + F_n$, with $F_0 = 0$ and $F_1 = 1$. These numbers are well-known for possessing amazing properties (consult book [4] to find additional references and history).

There are many papers in the literature which address Diophantine equations involving Fibonacci numbers. A long standing problem asking whether 0, 1, 8 and 144 are the only perfect powers in the Fibonacci sequence was recently confirmed by Bugeaud, Mignotte and Siksek [1]. Also, Hoggatt [2] conjectured that 1, 3, 21 and 55 are the only Fibonacci numbers in the triangular sequence, i.e., of the form $m(m+1)/2$. This was proved by Ming [14]. Many other similar equations have been considered during the years, we cite [5] and references therein for the state of the art of these kind of problems.

Here, our goal is to study the Diophantine equation

$$u_n = R(m)P(m)^{Q(m)}, \tag{1}$$

where $(u_n)_{n\geq 0}$ is a Lucas sequence and R, P and Q are integer polynomials (under some weak technical assumptions). The motivation of this kind of form comes from the study of tridiagonal matrices. For instance, the determinant of the $n \times n$ matrix

$$\begin{pmatrix} 2n & 1 & 0 & \cdots & 0 & 0 & 0 \\ n^2 & 2n & 1 & 0 & \cdots & 0 & 0 \\ 0 & n^2 & 2n & 1 & 0 & \cdots & 0 \\ \vdots & 0 & n^2 & 2n & \ddots & \ddots & \vdots \\ 0 & \vdots & \ddots & \ddots & \ddots & 1 & 0 \\ 0 & 0 & \cdots & 0 & n^2 & 2n & 1 \\ 0 & 0 & 0 & \cdots & 0 & n^2 & 2n \end{pmatrix}$$

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is equal to $n^n(n+1)$, for all $n \geq 1$.

In this paper, we shall show describe how a method based on p-adic valuations can settle this kind of equations. As our main result, we proved that

Theorem 1. *Let P, Q, R be integer polynomials with positive leading coefficients. Suppose that P and Q are non constant and that* $\deg P \leq \deg Q$. *Then there exist only finitely many solutions of the Diophantine equation* (1) *in positive integers m, n. Actually, all the solutions are effectively computable.*

We point out that the equation $u_n = P(m)$ was studied by many authors. We cite, for example, the result in [20, Theorem 1].

In the next theorem, we shall apply the method in the proof of Theorem 1 for finding an upper bound for the number of solutions when $u_n = F_n$, $R(m) = k(m+1)$ and $P(m) = Q(m) = m$. More precisely, we have

Theorem 2. *Let k be a given positive integer. If m, n are positive integers such that*

$$F_n = km^m(m+1), \tag{2}$$

then $m \leq \max\{89, \log k\}$.

Finally, we find all solutions when $1 \leq k \leq 50$.

Corollary 1. *The only solutions of the Diophantine equation in* (2) *for* $1 \leq k \leq 50$ *are*

$$(n, m, k) \in \{(3,1,1), (6,1,4), (9,1,17), (12,2,12)\}.$$

The proof of this corollary can be done by using *Mathematica* for the range $1 \leq m \leq 89$, $1 \leq k \leq 50$ and $1 \leq n \leq 849$.

2 Auxiliary results

Now, we recall some facts for the convenience of the reader.

Before stating the next lemma, we recall that for a positive integer n, the *order* (*or rank*) *of appearance* of n in the Fibonacci sequence, denoted by $z(n)$, is defined as the smallest positive integer k, such that $n \mid F_k$ (some authors also call it the *order of apparition*, as it was called by Lucas, or the *Fibonacci entry point*). There are several results on $z(n)$ in the literature. For example, recently, Marques [8–13] found closed formulas for this function for some sequences related to the Fibonacci and Lucas numbers.

We recall that the p-adic order (or valuation) of r, $\nu_p(r)$, is the exponent of the highest power of a prime p which divides r.

The p-adic order of Fibonacci numbers has been completely characterized, see [3, 7, 15]. For instance, from the main results of Lengyel [7], we extract the following result.

Lemma 1. *If $n \geq 1$ and $p \neq 2$ and 5, then*

$$\nu_p(F_n) = \begin{cases} \nu_p(n) + e(p), & \text{if } n \equiv 0 \pmod{z(p)}; \\ 0, & \text{if } n \not\equiv 0 \pmod{z(p)}. \end{cases}$$

Here $e(p) := \nu_p(F_{z(p)})$.

A proof of a more general result can be found in [7, pp. 236–237 and Section 5].

Actually, the p-adic valuation of a Lucas sequences was studied in [16] (see also [6]). In particular, it was proved in [16, Corollary 1.7] that

Lemma 2. *If $p \nmid b$ and $p \geq 5$, then*

$$\nu_p(u_n) = \begin{cases} \nu_p(n), & \text{if } p \mid \Delta; \\ \nu_p(n) + f(p), & \text{if } p \nmid \Delta, \tau(p) \mid n; \\ 0, & \text{if } p \nmid \Delta, \tau(p) \nmid n. \end{cases}$$

Here $\Delta = (\alpha - \beta)^2$ is the discriminant of $(u_n)_{n\geq 0}$ (in our case, $\Delta \neq 0$) and $f(p) = \nu_p(u_{\tau(p)})$. Also, $\tau(p) = \min\{n \geq 1 : p \mid u_n\}$ is the rank of apparition in the sequence $(u_n)_{n\geq 0}$.

For $(u_n)_{n\geq 0}$, we still have the existence of constants c and d, such that $\alpha^{n+c} \leq u_n \leq \alpha^{n+d}$, for all $n \geq 1$ (this can be proved by using the Binet formula and the fact that the roots of the recurrence satisfies that α/β is not a root of unity). We also have the following lemma

Lemma 3. *We have that $f(p) \leq 3p \log \alpha / \log p$, for all prime $p \geq d$.*

Proof. Since $p^{f(p)}$ divides $u_{\tau(p)}$, then

$$p^{f(p)} \leq u_{\tau(p)} \leq \alpha^{\tau(p)+d}.$$

Now, the Lemma 2.3 in [17] says that $\tau(p) \leq p + 1$ (in fact, it is proved that $\tau(p) \mid p - (-1)^{p-1}\left(\frac{\Delta}{p}\right)$, where $\left(\frac{\cdot}{p}\right)$ is the Legendre symbol). Thus

$$p^{f(p)} \leq u_{\tau(p)} \leq \alpha^{p+1+d} < \alpha^{3p}$$

which yields $f(p) \leq 3p \log \alpha / \log p$ as desired. □

Now we are ready to deal with the proof of the theorem.

3 Proofs

3.1 Proof of Theorem 1

Suppose that there are infinitely many solutions for the equation in (1). Let S be the set of the values of m belonging to a pair of a solution (n, m). Let k be the leading coefficient of Q and $\ell = \deg Q$. Also, let us denote by $T = \max\{T(P), T(Q), T(R)\}$, where $T(F(x)) = L(F) + \deg F$ (here $L(F)$ denotes the length of a polynomial F). Let $m \in S$, $m > m_0$, such that

$$k\frac{m^\ell}{4} > \frac{\log G(m)}{\log 7} + 1, \tag{3}$$

where $G(m) := T(m^T + 1)\frac{\log m}{\log \alpha} - c$ (observe that all sufficiently large integer in S must satisfies this inequality, since $\log G(m) = O(\log m)$).

Let $m \geq m_1 > m_0$ be an integer such that $Q(m) > km^\ell/2$. Also, let $\mathbb{P} = \{p_1, p_2, \ldots\}$ be the set of primes in increasing order. Set $p_r = \min\{s \in \mathbb{P} : 3L(P) \log \alpha / \log s < k/4\}$. As usual, let us denote $\rho(q)$ as the greatest prime factor of q. Let us suppose that P has at least two distinct roots (the other case can be handled in much the same way, by choosing a prime p which is a factor of $P(m) = t_1(m - b)^{t_2}$ for infinitely many m's). Since P has at least two distinct roots, then a result of Siegel [19] says that $\rho(P(m)) \to \infty$ as $m \to \infty$ (in fact $\rho(P(m)) \gg \log\log m$, see [18] and references therein). Thus $\rho(P(m)) \in \{p_1, \ldots, p_r\}$ only for finitely many values of m. So, take a value of $m > m_1$ belonging to S with $P(m) \neq 0$ and such that there exists a prime $p > \max\{5, b, d, p_r\}$ with $p \mid P(m)$. Now, we use Lemma 2 to apply the p-adic valuation in the relation in (1) to obtain

$$\nu_p(n) + f(p) \geq \nu_p(u_n) = \nu_p(R(m)) + Q(m)\nu_p(P(m)) \geq Q(m).$$

Thus $\nu_p(n) \geq Q(m) - f(p) > km^\ell/2 - f(p)$. By Lemma 3 , we have $f(p) \leq 3p \log \alpha / \log p$. Since $p \mid P(m)$, then $p \leq |P(m)| \leq L(P)m^{\deg P} \leq L(P)m^\ell$. Also, since $p > p_r$, then $3L(P) \log \alpha / \log p < k/4$. In conclusion,

$f(p) < km^{\ell}/4$ and so $\nu_p(n) > km^{\ell}/4$. Therefore, we have that $p^{\lfloor km^{\ell}/4 \rfloor} \mid n$. So

$$7^{km^{\ell}/4-1} \leq p^{\lfloor km^{\ell}/4 \rfloor} \leq n. \tag{4}$$

Since $\alpha^{n+c} \leq u_n = R(m)P(m)^{Q(m)} \leq m^{T(m^T+1)}$, for $m > 1$, then

$$n \leq T(m^T+1)\frac{\log m}{\log \alpha} - c = G(m). \tag{5}$$

By combining (3), (4) and (5), we arrive at the following absurdity

$$\frac{\log G(m)}{\log 7} + 1 < k\frac{m^{\ell}}{4} \leq \frac{\log n}{\log 7} + 1 \leq \frac{\log G(m)}{\log 7} + 1.$$

This shows that S must be finite and the proof is complete. □

3.2 Proof of Theorem 2

For the Diophantine equation in (2), we can suppose that $m > 2$. Then, there exists a prime number p dividing m. Thus p divides F_n and then

$$\nu_p(n) + 2e(p) \geq \nu_p(F_n) \geq m.$$

Thus $\nu_p(n) \geq m - 2e(p)$. Since $e(p) \leq p\log\phi/\log p$ (the proof is similar to the one of Lemma 3), then $e(p) < 0.44m$, for $p > 2$, and then $\nu_p(n) > 0.12m$ (here $\phi = (1+\sqrt{5})/2$ and we used that $e(2) = 1$ and that $m - 2 > 0.12m$, for $m \geq 3$). This means that $2^{0.12m-1} \leq n$.

Suppose, towards a contradiction, that $m > \max\{89, \log k\}$, then $\phi^{n-2} \leq km^m(m+1)$ and so $n \leq ((\log m + 1)m + \log(m+1))/\log\phi + 2$ (here we used that $\log k < m$). Therefore

$$0.12m - 1 < \frac{\log(((\log m + 1)m + \log(m+1))/\log\phi + 2)}{\log 2}.$$

Thus $m \leq 89$ which gives an absurdity. In conclusion, we have that $m \leq \max\{89, \log k\}$ as desired. □

Data Availability The data used to support the findings of this study are included within the article. Data sharing not applicable to this article as no datasets were generated or analysed during the current study.

Conflicts of Interest The author declares that there is no conflict of interest regarding the publication of this paper.

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