symmetry

MDPI

Article

Algebraic Numbers as Product of Powers of Transcendental Numbers

Pavel Trojovský

Department of Mathematics, Faculty of Science, University of Hradec Králové, 500 03 Hradec Králové, Czech Republic; pavel.trojovsky@uhk.cz; Tel.: +42-049-333-2801

Received: 17 June 2019; Accepted: 4 July 2019; Published: 8 July 2019

Abstract: The elementary symmetric functions play a crucial role in the study of zeros of non-zero polynomials in $\mathbb{C}[x]$, and the problem of finding zeros in $\mathbb{Q}[x]$ leads to the definition of algebraic and transcendental numbers. Recently, Marques studied the set of algebraic numbers in the form $P(T)^{Q(T)}$. In this paper, we generalize this result by showing the existence of algebraic numbers which can be written in the form $P_1(T)^{Q_1(T)} \cdots P_n(T)^{Q_n(T)}$ for some transcendental number T, where $P_1, \ldots, P_n, Q_1, \ldots, Q_n$ are prescribed, non-constant polynomials in $\mathbb{Q}[x]$ (under weak conditions). More generally, our result generalizes results on the arithmetic nature of z^w when z and w are transcendental.

Keywords: Baker's theorem; Gel'fond–Schneider theorem; algebraic number; transcendental number

1. Introduction

The name "transcendental", which comes from the Latin word "transcenděre", was first used for a mathematical concept by Leibniz in 1682. Transcendental numbers in the modern sense were defined by Leonhard Euler (see [1]).

A complex number α is called algebraic if it is a zero of some non-zero polynomial $P \in \mathbb{Q}[x]$. Otherwise, α is transcendental. Algebraic numbers form a field, which is denoted by $\overline{\mathbb{Q}}$. The transcendence of e was proved by Charles Hermite [2] in 1872, and two years later Ferdinand von Lindeman [3] extended the method of Hermite's proof to derive that π is also transcendental. It should be noted that Lindemann proved the following, much more general statement: The number e^{α}, where α is any non-zero algebraic number, is always transcendental (see [4]). In 1900, Hilbert raised the question of the arithmetic nature of the power α^{β} of two algebraic numbers α and β (it was the seventh problem in his famous list of 23 problems, which he presented at the International Congress of Mathematicians in Paris). The complete solution to this problem was found independently by Gel'fond and Schneider (see [5], p. 9) in 1934. Their results can be formulated as the following theorem (the ideas of the Gel'fond–Schneider proof were used partially in, e.g., [6–8]).

Theorem 1. *The Gel'fond–Schneider Theorem: Let α and β be algebraic numbers, with $\alpha \neq 0$ and $\alpha \neq 1$, and let β be irrational. Then α^{β} is transcendental.*

The Gel'fond–Schneider Theorem classifies the arithmetic nature of x^y when both x, y are algebraic numbers (because x^y is an algebraic number when y is rational). Nevertheless, when at least one of these two numbers is transcendental, anything is possible (see Table 1 below).

Table 1. Possible results for the power x^y when x or y is transcendental.

Value of x	Class of Numbers	Value of y	Class of Numbers	Power x^y	Class of Numbers
2	algebraic	$\log 3/\log 2$	transcendental	3	algebraic
2	algebraic	$i\log 3/\log 2$	transcendental	3^i	transcendental
e^i	transcendental	π	transcendental	-1	algebraic
e	transcendental	π	transcendental	e^π	transcendental
$2^{\sqrt{2}}$	transcendental	$\sqrt{2}$	algebraic	4	algebraic
$2^{\sqrt{2}}$	transcendental	$i\sqrt{2}$	algebraic	4^i	transcendental

In all the previous examples we have $x \neq y$ (in fact, we used the fact that the logarithm function is the inverse of the exponential function many times). Also, in the cases in which x and y are both transcendental (in the previous table), these numbers are possibly (though it's not proved) algebraically independent. So, what happens if we consider numbers of the form x^x with x transcendental? Is it possible that some of these numbers are algebraic? We remark that the numbers e^e and π^π are expected (but not proved) to be transcendental. In fact, it is easy to use the Gel'fond–Schneider Theorem to prove that every prime number can be written in the form T^T for some transcendental number T (for a more general result, see [9]). In this direction, a natural question arises: Given arbitrary, non-constant polynomials $P, Q \in \mathbb{Q}[x]$, is there always a transcendental number T such that $P(T)^{Q(T)}$ is algebraic? Note that $P(T)$ and $Q(T)$ are algebraically dependent transcendental numbers (so they do not come from our table). Marques [10] showed that the answer for the previous question is yes. More generally, he proved that for any fixed, non-constant polynomials $P(x), Q(x) \in \mathbb{Q}[x]$, the set of algebraic numbers of the form $P(T)^{Q(T)}$, with T transcendental, is dense in some connected subset of either $\mathbb{R}$ or $\mathbb{C}$. A generalization of this result for rational functions with algebraic coefficients was proved by Jensen and Marques [11]. However, the previous results do not apply, e.g., to prove the existence of algebraic numbers which can be written in the form $(T^2+1)^T \cdot T^{T^2+T+1}$, with T transcendental.

In this paper, we will solve this kind of problem completely by proving a multi-polynomial version of the previous results. The following theorem states our result more precisely.

Theorem 2. *Let $P_1, \ldots, P_n, Q_1, \ldots, Q_n \in \mathbb{Q}[x]$ be non-constant polynomials, such that the leading coefficients of the Q_j's have the same sign. Then the set of algebraic numbers of the form $P_1(T)^{Q_1(T)} \cdots P_n(T)^{Q_n(T)}$, with T transcendental, is dense in some open subset of the complex plane. In fact, this dense set can be chosen to be $\{Q(1+\sqrt[p]{2}) : Q \in K\}$, for some dense set $K \subseteq \mathbb{Q}(\sqrt{-1})\backslash\{0\}$, $K \cap \mathbb{Q} = \emptyset$, and any prime number $p > 2 \cdot (\max_{1\leq j\leq n}\{\deg Q_j\})!$.*

The proof of the above theorem combines famous classical theorems concerning transcendental numbers (like the Baker's Theorem on linear forms in logarithms and the Gel'fond–Schneider Theorem) and certain purely field-theoretic results. We point out that, in a similar way, we can prove Theorem 2 for rational functions with algebraic coefficients, but we choose to prove this simpler case in order to avoid too many technicalities, which can obscure the essence of the main idea.

2. Proof of Theorem 2

2.1. Auxiliary Results

Before we proceed to the proof of Theorem 2, we will need the following three lemmas. The first two lemmas come from the work of Baker on linear forms of logarithms of algebraic numbers (see [5], Chapter 2):

Lemma 1 (*Cf. Theorem 2.4 in [5]*)**.** *If $\alpha_1, \alpha_2, \ldots, \alpha_n$ are algebraic numbers other than 0 or 1, $\beta_1, \beta_2, \ldots, \beta_n$ are algebraic with $1, \beta_1, \beta_2, \ldots, \beta_n$ linearly independent over $\mathbb{Q}$, then $\alpha_1^{\beta_1}\alpha_2^{\beta_2}\cdots\alpha_n^{\beta_n}$ is transcendental.*

Lemma 2 (*Cf. Theorem 2.2 in [5]*)**.** *Any non-vanishing linear combination of logarithms of algebraic numbers with algebraic coefficients is transcendental.*

Let $\mathcal{F}$ be a family of polynomials. Hereafter, we will denote by $\mathcal{R}_{\mathcal{F}}$ the set of all the zeros of the polynomials in $\mathcal{F}$. The last of these lemmas is a purely field-theoretical result.

Lemma 3. *Let n be any positive integer and let $\mathcal{F}$ be a family of polynomials in $\mathbb{Q}[x]$ for which there exists a positive integer ℓ such that all polynomials in $\mathcal{F}$ have degree at most ℓ. Then for all prime numbers $p > \ell!$, the following holds:*

$$(1 + \sqrt[p]{2})^n \notin \mathbb{Q}(\mathcal{R}_{\mathcal{F}}). \tag{1}$$

Proof of Lemma 3. Set $\mathcal{F} = \{F_1, F_2, \ldots\}, K_n = \mathbb{Q}(\mathcal{R}_{F_1 \cdots F_n})$ and $t_n = [K_n : \mathbb{Q}]$. Since $K_n \subseteq K_{n+1}$, then $t_{n+1} = \ell_n t_n$, for some positive integer ℓ_n. Note that $\ell_n = [K_{n+1} : K_n] = [K_n(\mathcal{R}_{F_{n+1}}) : K_n] \leq (\deg F_{n+1})! \leq \ell!$. We claim that $(1 + \sqrt[p]{2})^n \notin \mathbb{Q}(\mathcal{R}_{\mathcal{F}})$ for all integers $n \geq 1$. For the contrary, there exist positive integers m and s such that $(1 + \sqrt[p]{2})^m \in K_s$. Then the degree of $(1 + \sqrt[p]{2})^m$ (which is p) divides t_s. However, $t_s = \ell_{s-1} \cdots \ell_1 t_1$ and $p > \ell! \geq \max_{j \in [1, s-1]}\{\ell_j, t_1\}$, which gives an absurdity. This completes the proof. $\square$

With these lemmas in hand, we can proceed to the proof of our main outcome.

2.2. The Proof

In order to simplify our presentation, we use the familiar notation $[a, b] = \{a, a + 1, \ldots, b\}$, for integers $a < b$.

Of course, it is enough to prove our theorem for the case that $P_1, \ldots, P_n$ are multiplicatively independent. For that, we take an open, simply connected subset Ω of $\mathbb{C}$, such that $P_j(x) \notin \{0, 1\}$ for all $x \in \Omega$ and $j \in [1, n]$. Choosing, for example, the principal branch of the multi-valued logarithm function, the function $f(x) := \prod_{j=1}^{n} P_j(x)^{Q_j(x)}$ is well defined and analytic in Ω. Moreover, $f(x)$ is a non-constant function. In fact, if f were constant then $f'(x) = 0$ in Ω and so

$$\sum_{j=1}^{n} Q_j'(x) \log P_j(x) + \sum_{j=1}^{n} \frac{Q_j(x) P_j'(x)}{P_j(x)} = 0, \tag{2}$$

for all $x \in \Omega$. We claim that $g(x) := \sum_{j=1}^{n} Q_j(x) P_j'(x) / P_j(x)$ is not the zero function in Ω. In fact, otherwise $G(x) := P_1(x) \cdots P_n(x) g(x)$ would be the zero polynomial, but the formal polynomial G has degree $\leq t := \max_{j \in [1,n]}\{m_1 + \cdots + m_n + t_j - 1\}$, where for all $j \in [1, n]$, m_j and t_j are the degree of P_j and Q_j, respectively. Now, if $t_{i_1} = \cdots = t_{i_s} = \max_{j \in [1,n]}\{t_j\}$, we get the relation $\sum_{j=1}^{s} m_{i_j} b_{i_j} = 0$ (the coefficient of x^t in G must be zero), where for all $j \in [1, n]$, b_j is the leading coefficient of Q_j. However $\sum_{j=1}^{s} m_{i_j} b_{i_j} \neq 0$, since $m_j > 0$ and b_j have the same sign. This gives a contradiction. Thus, there exists $\beta \in \Omega \cap \overline{\mathbb{Q}}$ such that $g(\beta) \neq 0$. Substituting then $x = \beta$ in (2), we have that $\sum_{j=1}^{n} Q_j'(\beta) \log P_j(\beta)$ is a nonzero algebraic number which contradicts Lemma 2. Hence f is a non-constant function.

Since f is a non-constant analytic function and Ω is an open connected set, $f(\Omega)$ is an open connected subset of $\mathbb{C}$. Let $\mathcal{F}$ be the family of polynomials $\{Q_i(x) - d : i \in [1, n],\ d \in \mathbb{Q}\} \cup \{x^2 + 1\}$. Clearly, each polynomial in $\mathcal{F}$ has degree $\leq 2\ell := 2\max\{\deg Q_1, \ldots, \deg Q_n\}$. Thus, the conditions to apply Lemma 3 are fulfilled. Hence, for $p > 2\ell!$, we have that the set $\mathcal{P} := \{r(1 + \sqrt[p]{2}) : r \in \mathbb{Q}(\sqrt{-1}) \backslash \{0\}\}$ forms a dense subset of $\mathbb{C}$ and no positive integer power of its elements lies in $\mathbb{Q}(\mathcal{R}_{\mathcal{F}})$. Since $f(\Omega)$ is open, $f(\Omega) \cap \mathcal{P}$ is dense in $f(\Omega)$. Now, it remains to prove that every number in this intersection can be written in the desired form. For that, let $\alpha := r(1 + \sqrt[p]{2}) \in f(\Omega) \cap \mathcal{P}$, then

$$\alpha = f(T) = \prod_{j=1}^{n} P_j(T)^{Q_j(T)}, \tag{3}$$

where $T \in \Omega$. Therefore, it is enough to prove that T is a transcendental number. To get a contradiction, suppose the contrary; i.e., that T is algebraic. Then $P_1(T), \ldots, P_n(T), Q_1(T), \ldots, Q_n(T)$ are also algebraic numbers. By the choice of Ω, Lemma 1 ensures the existence of a nontrivial $\mathbb{Q}$-relation among $1, Q_1(T), \ldots, Q_n(T)$ (this implies, in particular, that the degree of T is at most ℓ). Without loss of generality we can assume that $a_n Q_n(T) = a_0 + \sum_{j=1}^{n-1} a_j Q_j(T)$, where a_j is an integer, with $a_n > 0$. Therefore, identity (3) becomes

$$\alpha^{a_n} = P_n(T)^{a_0} \left(P_1(T)^{a_n} P_n(T)^{a_1}\right)^{Q_1(T)} \cdots \left(P_{n-1}(T)^{a_n} P_n(T)^{a_{n-1}}\right)^{Q_{n-1}(T)}.$$

Note that $\alpha^{a_n} P_n(T)^{-a_0}$ is an algebraic number and $P_j(T)^{a_n} P_n(T)^{a_j} \neq 0$ for $j \in [1, n-1]$. We claim that $P_j(T)^{a_n} P_n(T)^{a_j} \neq 1$ for some $j \in [1, n-1]$. In fact, otherwise we would have $\alpha^{a_n} = P_n(T)^{a_0} \in \mathbb{Q}(T)$ and so $(1 + \sqrt[p]{2})^{a_n} \in \mathbb{Q}(T, \sqrt{-1})$ has degree at most 2ℓ. However, this gives an absurdity since the degree of $(1 + \sqrt[p]{2})^{a_n}$ is $p > 2\ell!$. Thus, sometimes $P_j(T)^{a_n} P_n(T)^{a_j}$ is an algebraic number different from 0 and 1, so we can apply Lemma 1 again to get a $\mathbb{Z}$-relation $b_{n-1} Q_{n-1}(T) = b_0 + \sum_{j=1}^{n-2} b_j Q_j(T)$, where b_j is an integer, with $b_{n-1} > 0$. Analogously, one can iterate this process $n-1$ times to conclude that

$$\alpha^q = A(T) \left(P_1(T)^{c_1} \cdots P_n(T)^{c_n}\right)^{Q_1(T)}, \tag{4}$$

where $A(T) \in \mathbb{Q}(P_1(T), \ldots, P_n(T))$ and q, c_j's $\in \mathbb{Z}$, with $q > 0$. If $\prod_{j=1}^n P_j(T)^{c_j} = 1$, we would arrive at the same absurdity as before since $\mathbb{Q}(P_1(T), \ldots, P_n(T)) \subseteq \mathbb{Q}(T)$. Thus $\prod_{j=1}^n P_j(T)^{c_j} \in \overline{\mathbb{Q}} \backslash \{0, 1\}$, so by the Gel'fond–Schneider Theorem we deduce that $Q_1(T)$ is a rational number, say r/s, with some integers r and s, $s > 0$. Hence, T belongs to $\mathcal{R}_{Q_1(x)-r/s} \subseteq \mathcal{R}_{\mathcal{F}}$. But then $\alpha^{qs} = A(T)^s P_1(T)^{rc_1} \cdots P_n(T)^{rc_n}$ (see (4)) and thus $(1 + \sqrt[p]{2})^{qs} \in \mathbb{Q}(\mathcal{R}_{\mathcal{F}})$, contradicting the choice of p in Lemma 3. In conclusion, T must be transcendental, and this completes the proof. □

3. Conclusions

In this paper, we use analytic (complex analysis), algebraic (Galois' extensions and symmetry) and transcendental tools (Baker's theory) to prove, in particular, the existence of infinitely many algebraic numbers of the form $P_1(T)^{Q_1(T)} \cdots P_n(T)^{Q_n(T)}$, where T is a transcendental number and $P_1, \ldots, P_n, Q_1, \ldots, Q_n$ are previously fixed rational polynomials (under some weak technical conditions).

Funding: The author was supported by Project of Excelence PrF UHK, University of Hradec Králové, Czech Republic 01/2019.

Acknowledgments: The author thanks anonymous referees for their careful corrections and their valuable comments that helped to improve the paper's quality and readability.

Conflicts of Interest: The author declares no conflict of interest.

References

1. Erdös, P.; Dudley, U. Some remarks and problems in number theory related to the work of Euler. *Math. Mag.* **1983**, *56*, 292–298. [CrossRef]
2. Hermite, C. Sur la fonction exponentielle. *C. R. Acad. Sci. Paris I* **1873**, *77*, 18–24.
3. Lindemann, F. Über die Zahl π. *Math. Ann.* **1882**, *20*, 213–225. [CrossRef]
4. Marques, D. On the arithmetic nature of hypertranscendental functions at complex points. *Expo. Math.* **2011**, *29*, 361–370. [CrossRef]
5. Baker, A. *Transcendental Number Theory*; Cambridge Mathematical Library, Cambridge University Press: Cambridge, UK, 1990.
6. Bundschuh, P.; Waldschmidt, M. Irrationality results for theta-functions by Gel'fond–Schneider method. *Acta Aritmetica* **1989**, *53*, 289–307
7. Srivastava, H.M.; Araci, S.; Khan, W.A.; Acikgoz, M. A Note on the Truncated-Exponential Based Apostol-Type Polynomials, *Symmetry* **2019**, *11*, 538. [CrossRef]

8. Waldschmidt, M. Auxiliary functions in transcendental number theory, *Ramanujan J.* **2009**, *20*, 341–373. [CrossRef]
9. Sondow, J.; Marques, D. Algebraic and transcendental solutions of some exponential equations. *Ann. Math. Inform.* **2010**, *37*, 151–164.
10. Marques, D. Algebraic numbers of the form $P(T)^{Q(T)}$, with T transcendental. *Elem. Math.* **2010**, *65*, 78–80. [CrossRef]
11. Jensen, C.U.; Marques, D. Some field theoretic properties and an application concerning transcendental numbers. *J. Algebra Appl.* **2010**, *9*, 493–500. [CrossRef]