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A Modification of the Vehicle Routing Problem

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Abstract: There are many modifications of the standard vehicle routing problem (VRP). VRP consists in optimization of vehicle routes which contain given set of nodes. The matrix of node distances and demand of goods are specified. In VRP proposed in the paper nodes are suppliers (shops, manufactures, producers…) with different prices of offered goods. The price list of goods of producers is given and the distance matrix of nodes too. The first node is depot of vehicles to which goods are delivered. In the first node is located a buyer with a given list of demanded quantity of goods. Buyer has to minimize total costs which is sum of transportation costs and costs of purchased goods (which depends on suppliers' nodes). Goal is to find optimal routes though nodes which are supplier of some purchased goods. The mathematical model is proposed which is demonstrated by numerical example. In addition to the model a heuristic method is shown.

Keywords: vehicle routing problem; integer programming; heuristic method

JEL Classification: C44

1. Introduction

Traditional formulation of vehicle routing problem (VRP) assumes n nodes, where the first node represents the depot and the remaining ones the customers. The merchandise is transported using the routes starting and ending in the depot. The transport itself is realized by the vehicles with certain capacity and the customer-node requests are given by the volume used for the containment of the requested merchandise in a given vehicle. The route length depends on the order of the nodes of a given route and can be calculated using the distance matrix between each pair of nodes. Instead of the node distance one can also calculate the transport cost of a given vehicle from one node to another. The aim is to minimize the total sum of route distances or eventually to minimize the transport cost of the routes using given vehicles.

The solution procedure of VRP must ensure the following two conditions:

a) All nodes are included at least in one of the routes.

b) The sum requests of all nodes of a route must not exceed the capacity of the vehicle for this route.

This problem can be formulated as integer linear programming model and solved using appropriate software tools. VRP and in general, linear integer programming problems, belong among NP hard problems, i.e. if the number of nodes is higher, in reality more than approx. 30 nodes, it is impossible to obtain optimal solution using standard LP integer solvers (branch and bound method) in a reasonable time. Except for mathematical models one can use heuristic methods such as nearest neighborhood method, insert method or savings method, which can help to obtain a suboptimal solution in reasonable time.

There are many modifications of the conventional form of VRP, which arise as a results of merchandise transport in praxis (Laporte, 1992). The following ones belong among the most interesting ones – VRP involving vehicles with different capacities and transport cost; VRP with more than one depot; split delivery VRP; VRP involving stochastic demand in nodes; VRP with time windows, where the time of vehicle arrival in the node must be inside to a certain time interval denoted as time window (Braysy & Gendreau, 2005; Desrochers et al., 1992), and others.

The traditional VRP problem is described in the literature enough, heuristic methods are proposed, and a survey of these approaches is summarized in (Laporte, 1992).

2. Problem Formulation and Mathematical Model

In the vehicle routing problem studied in this paper there are n nodes where first node is depot. The first node is depot of vehicles to which goods are delivered. In the first node is located a buyer with a given list of demanded quantity of goods. Buyer has to minimize total costs which is sum of transportation costs and costs of purchased goods (which depends on suppliers' price list). Other nodes are suppliers (shops, manufactures, producers…) with different prices of offered goods. The price list of goods of producers is given and the distance matrix of nodes too. Buyer (in the first node) has to minimize total costs which is sum of transportation costs and costs of purchased goods (which depends on suppliers' price list). Goal is to find optimal routes though nodes which are supplier of some purchased goods. The optimal solution contains a decision which product is purchased from certain supplier.

The mathematical model assumes the demanded product to be available in every node. In case there is a product that is not available in a given node, we set the product price to be prohibitively high.

Parameters of the model:

n number of nodes, node 1 is depot,

d_{ij} vehicle transport costs from node i to node j,

m number of products: $P_1, P_2, \ldots, P_m$,

q_k customer demand for k-th product,

c_{ki} price per unit of the k-th product in the i-th node,

W vehicle capacity,

$M > 0$ prohibitive value.

Variables of the model:

x_{ij} binary variable, equals 1 if the vehicle travels from node i to node j, where $i \neq j$,

y_{ik} binary variable, equals 1 if product k is loaded in node i,

z_i binary variable, equals 1 if the vehicle visits node i,

u_i anticyclic conditions variable in (6).

Objective function (1) is defined as a sum of the transport costs and the product costs. Function (1) is being minimized. Equation (2) means that if the node i is entered then z_i=1, et vice versa. Condition (3) assures that once the vehicle enter a node it will have to leave it. Equation (4) prescribes that the k-th product has to be picked up in one of the nodes. If the product is picked up in the i-th node the vehicle has to pass through (5). Conditions (6) and (7) state that the vehicle capacity will not be exceeded. At the same time (6) prevents partial cycles.

The mathematical model (1) – (8) can be modified for the case of split demand, i.e. the demand of the k-th product q_k can be divided into several fractions and these fractions are transported by different vehicles from different nodes.

Mathematical model:

$$f(x) = \sum_{i=1}^{n} \sum_{j=1}^{n} d_{ij} x_{ij} + \sum_{i=2}^{n} \sum_{k=1}^{m} c_{ki} q_k y_{ik} \quad \rightarrow min \tag{1}$$

$$\sum_{j=1}^{n} x_{ij} = z_i , \qquad i = 2,3, \dots , n, \tag{2}$$

$$\sum_{i=1}^{n} x_{ij} = \sum_{i=1}^{n} x_{ji} , \qquad i = 1,2, \dots , n, \tag{3}$$

$$\sum_{i=2}^{n} y_{ik} = 1 , \qquad k = 1,2, \dots , m, \tag{4}$$

$$\frac{1}{M} \sum_{k=1}^{m} y_{ik} \le z_i , \quad i = 2,3, \dots , n, \tag{5}$$

$$u_i + \sum_{k=1}^{m} q_k y_{jk} - W(1 - x_{ij}) \le u_j , \qquad i = 1,2, \dots , n, j = 2,3, \dots , n, i \ne j, \tag{6}$$

$$u_j \le W , \ j = 2,3, \dots , n, \tag{7}$$

$$x_{ij} \ is\ binary, i, j = 1,2, \dots , n, i \ne j , \tag{8}$$

$$y_{ik} \ is\ binary \ i = 2,3, \dots , n, k = 1,2, \dots , m ,$$

$$u_i \ge 0, \ x_{ii} = 0, \ i = 1, 2, \dots , n.$$

3. Numerical Example

The number of nodes is 11, whereas node 1 is the depot. The vehicle capacity is W=100. The aim is to pick up and transport five products $P_1, P_2, \dots, P_5$ in amount $\boldsymbol{q}$ = (24, 35, 42, 20, 45) to the node 1.

The product unit costs in each node are given in Table 1.

Table 1. Product costs

c_{kj}	j=2	3	4	5	6	7	8	9	10	11
k=1	18	10	13	8	12	5	4	13	2	3
2	19	15	14	6	12	5	4	9	5	2
3	17	12	15	7	18	4	2	11	5	6
4	19	15	18	6	9	3	5	4	14	5
5	8	19	2	4	6	3	5	4	6	4

Table 2. Transport cost matrix d_{ij}

0	13	6	55	93	164	166	168	169	241	212
13	0	11	66	261	175	177	179	180	239	208
6	11	0	60	97	168	171	173	174	239	209
55	66	60	0	82	113	115	117	117	295	265
93	261	97	82	0	113	115	117	118	333	302
164	175	168	113	113	0	6	4	2	403	374
166	177	171	115	115	6	0	8	7	406	376
168	179	173	117	117	4	8	0	3	408	378
169	180	174	117	118	2	7	3	0	409	379
241	239	239	295	333	403	406	408	409	0	46
212	208	209	265	302	374	376	378	379	46	0

We will decide in which node each product is purchased so that total costs, i.e. sum of the transport and the product cost, is minimized.

Result of the mathematical model (1)-(8) are two routes 1-4-3-1 and 1-7-8-1 with travel costs 463 units. Product P_1 is picked up in node 3, product P_5 in node 4, product P_4 in node 7 and both products P_2 and P_3 in node 8. The product costs are equal to 614 units and considering the transport costs, the total costs amount to 463 + 614 = 1,077 units. Those final costs provide the minimal value.

4. Heuristic Method

Due to NP hardness of VRP it needs to be proposed heuristic methods. Next, a heuristic method is formulated and also illustrated by an example.

We will create the routes in the form $1 - i_1 - i_2 - \cdots - i_s - 1$. In the first step we will find the first node i_1, in the second step we will insert other nodes i_2, i_3,…i_s in the initial route $1 - i_1 - 1$.

We denote a set ***PP*** the set of products which were not assured by the routes created so far. In the beginning of the heuristic we set $PP = P = \{P_1, P_2,\ldots,P_m\}$. We apply step 1 and step 2 to all routes until all products are covered, i.e. set PP is empty. Costs of the product P_k purchased at the node i we will denote $pc_{ik} = c_{ik} * q_k$.

Gradually we create routes until all products are assigned to some route node, i.e. the set PP is empty.

Route creation:

Step 1: *{the first node i1 choice}*

Value $\Delta_{i1} = min_{i=2,3,\ldots,n} \Delta_i$, where $\Delta_i = \sum_{\substack{k=1 \\ k \in PP}}^{m} pc_{ik} + d_{1i} + d_{i1}$, $i = 2,3,\ldots,n$, means costs if all products from PP are purchased only at this node $i1$ and these costs are minimal over set of nodes. The first node of the route is denoted *i1*. Due a limited capacity of vehicle it can be purchased only a part of *PP* at node *i1*. Next, we will choose those products from *PP* which are the cheapest and capacity of vehicle is not exceeded. The set of these products is denoted *PF* ⊂ *PP*. Put *PP:= PP - PF.*

Step 2. *{insert other nodes for products from PF}*

For product P_k from *PF*:

Let $\rho_r = min_{i=2,3,\ldots,n}\, cp_{i1,k} - cp_{i,k'i'}$ for r = 2,3,…,*n*. If $\rho_r < 0$, then product costs of P_k are cheaper at node *r* than at node i_1. But, by inserting the node *r* in route increase transportation costs of the route. So, we try insert node *r* such a way to minimize an increase of transportation costs. Let σ_r is the minimal increase of transportation costs by inserting the node *r* in the route. Total costs will decrease only if $\rho_r + \sigma_r$ is negative. We insert the node *r* in the route if $(\rho_r + \sigma_r)$ is minimal for all nodes *r* and at the same time is negative.

5. Example (Continuation of the Section 3)

There are 11 nodes and 5 products. Distance matrix is in Table 2, product unit costs see Table 1 and quantity of products $\boldsymbol{q}$ = (24, 35, 42, 20, 45). Capacity of vehicle is *V*=100. Put *PP = {P₁, P₂,…,P₅}*. Costs of the product P_k purchased at the note i we will denote $pc_{ik} = c_{ik} * q_k$ and are shown in Table 3.

Table 3. Value $pc_{ik} = c_{ik} * q_k$

Node	2	3	4	5	6	7	8	9	10	11
P_1	432	240	312	192	288	120	72	312	120	96
P_2	665	525	490	210	420	175	140	315	175	70
P_3	765	540	675	315	810	180	90	495	225	270
P_4	380	300	360	120	180	80	100	60	280	100
P_5	360	855	90	180	270	135	225	180	270	180

Route 1:

Step 1:

Minimal value of Δ_i is Δ_8=963 (see Table 4) then the node i_1=8, so we have initial route 1 – 8 – 1. Transportation costs of this route are $d_{18}+d_{81}$= 168 + 168 = 336.

Table 4. Value $\Delta_i = \sum_{k=1, k \epsilon PP}^{5} pc_{ik} + d_{1i} + d_{i1}$

Node i	2	3	4	5	6	7	8	9	10	11
Δ_i	2,628	2,462	2,037	1,203	2,296	1,022	**963**	1,700	1,552	1,140

The capacity of vehicles is *V*=100, therefore we have to pick up the products by descending value $pc_{ik} = c_{ik} * q_k$ not to exceed capacity of vehicle *V*. Table 5 contains chosen products with their costs and volume. Transportation costs route 1 are 336, costs of products P_1, P_2 and P_4 assured by this route are 262 (see Table 5). Total costs are 336+262=598. Put *PF = {P₁, P₃, P₄}*. *PP = PP - PF ={P₂, P₅}*. In step 2 we try decrease the costs of route 1.

Table 5. Products assured in the route 1

Product	Costs	Volume
P_1	72	24
P_3	90	45
P_4	100	20
Sum	262	89

Step 2:

Product costs of P_4 are cheaper at node 7 and node 9 than at node i_1=8. Decrease of product costs at node 7 are 20, decrease at node 9 are 40 (see Table 3). By inserting node 7 in route 1 increase transportation costs by 6, by inserting node 9 in route 1 increase transportation costs by 4 (see Table 2). Inserting node 7 decrease total costs (transportation plus product costs) by 20-6=14, inserting node 9 decrease is 40-4=36. Finally, we insert node 9. So, route 1 is in the form 1 – 9 – 8 – 1. Route assures products P_1, P_3 and P_4. Transportation costs of the route are 340, product costs are 222, total costs are 340 + 222 = 562.

Route 2:

Step 1:

It remains to ensure products $PP = \{P_2, P_5\}$.

Table 6. Value $\Delta_i = \sum_{\substack{k=1 \\ k \epsilon PP}}^{5} pc_{ik} + d_{1i} + d_{i1}$ is product costs for products P_2 and P_5

Node	2	3	4	5	6	7	8	9	10	11
Δ_i	1051	1392	690	**576**	1018	642	**701**	833	927	674

A node with cheapest costs of both products P_2 and P_5 is node 5 (see Table 6). Therefore, second route will be 1 – 5 – 1 and the capacity of the vehicle will not be exceeded. Transportation costs are 186 and product costs are 390. Total costs are 186 + 390 = 576.

Step 2:

Product P_5 is cheapest at the node 5, but for the product P_2 are lover costs in the nodes 7, 8, 10 and 11 (see Table 3). Changes in costs by insertion another node in the route are shown in Table 7. Because by inserting each of the nodes 7, 8, 10, 11 they will rise total costs, no node will be inserted in the route 1 – 5 – 1. This is stop of the method.

Result is two routes:

Route 1: 1 – 9 – 8 – 1, with costs 562.

Route 2: 1 - 5 – 1, with costs 576.

Total costs for both routes are 562 + 576 = 1138.

Table 7. Changes in costs by insertion another node in the route

Node	Transportation costs	Product costs	Total costs
7	+95	-35	+60
8	+99	-70	+29
10	+388	-35	+353
11	+328	-140	+188

6. Conclusions

In this paper we have considered a new modification of vehicle routing problem. The mathematical model and heuristic method are proposed, on numerical example are both illustrated.

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