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Nonlinear Cournot Duopoly Game

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Abstract. The study of structure of different markets haven't been finished yet. Even such a well-known concept as oligopoly can be described by different models with different assumptions and with different values of parameters. The aim of this paper is to consider a nonlinear inverse demand function in Cournot duopoly model. Provided there is a sufficiently large proportion between costs of the two firm it is possible to observe nonlinear phenomena such as bifurcation of limit values of production or deterministic chaos. To prove a sensitive dependence on initial condition, that accompanies deterministic chaos, the concept of Lyapunov exponent is used. We also discuss that the particular values of parameters are not important for the discussion about the mentioned nonlinear phenomena but that their possible presence is worth to know.

Keywords: Bifurcation, Cournot Duopoly, Lyapunov Exponent, Nonlinear Model, Oligopoly.

1 Introduction

One of the effects of globalization is that there emerge very rich and powerful corporations in the economic world. Such, usually multinational, companies of considerable size can have sufficiently influence to determine prices at a particular market. In consequence of this structure a trade can be completely controlled by several firms. This is the reason why oligopoly structure markets and their different models are studied and modified again, [2, 9, 11, 7, 8]. It is well known that oligopoly markets consider a few producers that produce the same good or goods that are perfect substitutes. Each company must consider not only the demand of market, but also the actions of the competitors, the property that is known as interdependence. In this paper we restrict to the case of market with two companies that is called duopoly, [3]. In contrast of the classical Cournot duopoly game it is considered a nonlinear inverse demand function. These problems are studied at [10] and [6]. In this paper we consider similar properties as in [9].

2 Model

First we briefly remind a classical model of oligopoly, [5, 12], and in particular we present Cournot duopoly assumptions, [3]. Then we introduce a special nonlinear demand function, [10], that allows us to present some complex phenomena of duopoly game.

2.1 Fundamental Principles of Cournot Oligopoly Model

Let $n, n \in N_0$, be the number of companies at the given market. Denote $D = \{1,2,\dots,n\}$ a finite set and let C_i, $i \in D$, be the company that produce the homogeneous output $q_i(t)$ at the given time period t, $t \in N_0$. All companies make plans for their production $q_i(t+1)$ in the next time period in order to maximize their expected profit P_i or expected utility. Profit of each companies depends on the price $p(t+1)$ at which the good is sold in period $t+1$ and this price depends on the total supply $Q(t) = \sum_{j \in D} q_j(t)$ according to a given inverse demand function

$$p(t+1) = p_D[Q(t)]. \quad (1)$$

To simplify the further consideration we introduce the following notation

$$Q_i(t) = \sum_{j \in D \setminus \{i\}} q_j(t) = Q(t) - q_i(t), \quad (2)$$

that represents the total output of the rest of the industry expect for the firm C_i at the period t. Notice, that the relation $Q(t) = q_i(t) + Q_i(t)$ is valid for all $i \in D$. The profit P_i of company C_i can be now expressed as

$$P_i\big(q_i(t), Q_i(t)\big) = q_i(t) \cdot p_D[q_i(t) + Q_i(t)] - c_i(t) \cdot \big(q_i(t), Q_i(t)\big), \quad (3)$$

where $c_i(\cdot)$ is the cost function of company C_i. Moreover the production for the next period $t+1$ of the company C_i can be found as a solution of the following optimization problem

$$q_i(t+1) = arg\max_{x \in X_i} P_i(x, Q_i^e(t+1)), \quad (4)$$

where $Q_i^e(t+1)$ represents the total output of the rest of the industry expected by firm C_i for the next time period $t+1$ and X_i, $X_i \subset [0,\infty)$, is the strategy set which is used for selection of the optimal production of the company C_i. The principal assumption of the model is how to represent particular expectations about production of other companies. The original Cournot assumption is a simple naive expectation

$$Q_i^e(t+1) = Q_i(t) \quad (5)$$

for all $i \in D$.

2.2 Dynamics of Cournot Model

Let all problems (4) have theirs unique solutions. If (5) is applied we can write

$$q_i(t+1) = R_i(Q_i(t)), \qquad i \in D, \quad (6)$$

where $R_i\colon \prod_{j=1,j\neq i}^{n} X_j \to X_i$ is the reaction function of company C_i or the best response or best reply mapping of firm C_i. To study this dynamical problem in more details a more specific forms of inverse demand functions and costs functions are necessary.

2.3 Nonlinear Model

Similarly as in [9] we consider that (a) the quantity demanded is reciprocal to price and (b) the firms operate under constant unit costs. Particularly we assume that (1) has a form

$$p(t+1) = \frac{1}{Q(t)}, \qquad t \in N_0, \tag{7}$$

and instead of (3) it is possible to write profit function in the form

$$P_i(x, Q_i(t)) = \frac{x}{x + Q_i(t)} - a_i x, \qquad i \in D, \tag{8}$$

where $a_i, a_i > 0$, is a constant unit costs of the firm C_i and x substitutes $q_i(t)$ for a specific value of period t. Now it is possible to solve profit maximum problem (4). The first order conditions of this problem are

$$P_i'\big(x, Q_i(t)\big) = \frac{Q_i}{\big(x + Q_i(t)\big)^2} - a_i = 0, \ \ i \in D. \tag{9}$$

As the numerator, residual demand, as well the unit cost a_i, are positive, we can find their roots, and solve (9) for the simple reaction function

$$x = \sqrt{\frac{Q_i}{a_i}} - Q_i, \quad i \in D. \tag{10}$$

The reaction x is positive provided that

$$Q_i(t) < \frac{1}{a_i}, \quad i \in D. \tag{11}$$

If not, then the negative outcome has to be replaced by sufficiently small and nonzero outcome $\varepsilon, \varepsilon > 0$, which allows us to construct the resulting reaction functions for dynamics (6). For all $i \in D$ we put

$$q_i(t+1) = R_i(Q_i(t)) = \begin{cases} \sqrt{\dfrac{Q_i}{a_i}} - Q_i, & Q_i(t) < \dfrac{1}{a_i}, \\ \varepsilon, & Q_i(t) \geq \dfrac{1}{a_i}\cdot \end{cases} \tag{12}$$

This system of difference equations (12) stands for a particular dynamics (6) of the introduced nonlinear oligopoly model.

3 Results and Discussion

For the rest of this paper we will consider a special case of a duopoly game. In this case $n = 2$ and immediately from (2) we have

$$Q_i(t) = \begin{cases} q_2(t), & i = 1, \\ q_1(t), & i = 2. \end{cases} \tag{13}$$

If duopolists partially adjust their quantities towards the their best replies according to (6) and (13), the dynamical system is generated by the iteration of the map

$$F: (q_1(t+1), q_2(t+1)\ ) = (R_1(q_2(t)), R_2(q_1(t)))\ , \tag{14}$$

where $R_1: X_2 \to X_1$ and $R_2: X_1 \to X_2$ are reaction functions of companies C_1 and C_2 given by (12). If the initial conditions $(q_1(0), q_2(0)) \in X_1 \times X_2$ are given, a trajectory

$$\{(q_1(t), q_2(t))\}_{t=0}^{\infty} = \{F^t(q_1(0), q_2(0))\}_{t=0}^{\infty}, \tag{15}$$

is generated by t −th iteration F^t, $t \in N_0$, of map (14) and it produces Cournot tatonnement, [6].

3.1 Equilibrium and its properties

If there is a fixed point (q_1°, q_2°) of the map (14) it is called Cournot-Nash equilibrium, [5]. This equilibrium can be found as a solution to the system of equations

$$(q_1^\circ, q_2^\circ) = F(q_1^\circ, q_2^\circ). \tag{16}$$

With the particular form of reaction functions (12) it is possible to find the following nonzero equilibrium

$$(q_1^\circ, q_2^\circ) = \left(\frac{a_2}{(a_1+a_2)^2}, \frac{a_1}{(a_1+a_2)^2}\right). \tag{16}$$

The stability of this stationary point can be determined from the Jacobian matrix of map (14) enumerated at the stationary point, [4]. It is possible to find

$$J(q_1, q_2) = \begin{pmatrix} 0 & \dfrac{1}{2\sqrt{a_1 q_2}} - 1 \\ \dfrac{1}{2\sqrt{a_2 q_1}} - 1 & 0 \end{pmatrix}, \tag{17}$$

which means that at stationary point (16) we have

$$J(q_1^\circ, q_2^\circ) = \begin{pmatrix} 0 & \dfrac{a_2 - a_1}{2a_1} \\ \dfrac{a_1 - a_2}{2a_2} & 0 \end{pmatrix}. \tag{18}$$

The eigenvalues of matrix (18) are imaginary as follows

$$\lambda_1 = -i\,\frac{|a_1 - a_2|}{2\sqrt{a_1 a_2}}, \quad \lambda_2 = i\,\frac{|a_1 - a_2|}{2\sqrt{a_1 a_2}}. \tag{19}$$

The stationary point (q_1°, q_2°) is asymptotically stable if $|\lambda| < 1$, [4]. Solving this problem we can observe that such situation happens when one of the following relation for the ratios a_1/a_2 or a_2/a_1 of unit costs is valid

$$3 - 2\sqrt{2} < \frac{a_1}{a_2} < 3 + 2\sqrt{2} \ \text{ or } \ 3 - 2\sqrt{2} < \frac{a_2}{a_1} < 3 + 2\sqrt{2}. \tag{20}$$

As soon as the ratios of the unit costs fall outside these intervals the stationary point is not stable.

3.2 Bifurcation diagram

To study periodical points of map (14) it is convenient to construct a bifurcation diagram. It shows the relationship between values of a parameter and values of fixed points or values of periodic orbits of the given dynamical system. More generally asymptotically stable stationary points and periodical points are special types of attractors that can be briefly characterized in two steps as follows: (i) a limit set of a point $q \in W$, where W is an open set in R^2, is the set of all points $a \in W$, such that there exists a sequence $t_i \to \infty$ and $\lim_{t_i \to \infty} F^{t_i}(q) = a$; (ii) a compact set $A \subset W$ is called attractor if there is a neighborhood U of A such that A is the limit set of all initial values $q(0) \in U$. Simply put an attractor is a set of all points to which trajectories starting at initial points from a neighborhood of the set will converge, [1].

Now it is possible to characterize the bifurcation diagram in more details: if $\lambda, \lambda \in R$, is a parameter of dynamical system (14) and A_λ is a set of all attracting points for the given value of $\lambda \in M$, where M is the parameter set of our interest, then the bifurcation diagram is the graph of the relation $\{(\lambda, A_\lambda) |\, \lambda \in M\}$. This figure shows the birth, evolution and extinction of attracting sets, [1]. Because (14) depends symmetrically on two parameters a_1 and a_2, corresponding cost prices of firms, we consider the price ratio $\lambda = a_1/a_2$ as a bifurcation parameter and we also set $a_2 = 1$, as a price unit.

The algorithm for plotting a bifurcation diagram is based on the direct application of definition of attractor – instead of computing exact limit points only points for sufficiently large number of iterations of map (14) are considered. The algorithm can be described as follows: (i) choose the initial value λ of the parameter of map (14); (ii) at random choose an initial value $(q_1(0), q_2(0))$ of map (14); (iii) calculate a few first iteration od (14) and ignore them; (iv) calculate a few next iterations of (14) and plot them; (v) increment the value a of the parameter of map (14) and repeat above steps until you reach the end of the parameter sets. See also [1]. The given algorithm was implemented in Matlab and its result is reported at Fig. 1.

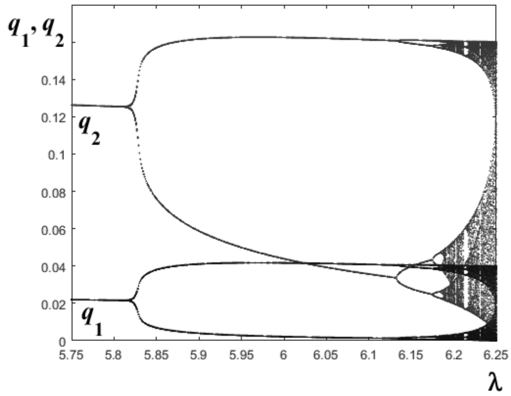

Fig. 1. Bifurcation diagram of map (14). The black diagram describes the dependence of limit points of variable q_1 on parameter λ and similarly the grey diagram describes limit points of variable q_2.

3.3 Lyapunov exponent

The bifurcation diagram can also point out to the phenomenon called deterministic chaos. In this case almost all intervals seem to be filled by the plot. Once, such a phenomenon is observed it is useful to compute Lyapunov exponents for special values of parameters. It is a method how to formally identify the sensitive dependence of the given system on initial conditions, which is one of the characteristic attribute of chaotic behavior. This exponent measures the exponential rate of separation of very close trajectories.

Here we give only a concise characterization based on [1]. Let F be a smooth map on R^2, similarly as in (14), and let $J_n = DF^n(v_0)$, where v_0 is an initial point and D is the first derivative of the map F. In other words J is a Jacobian of the map F. For $k \in \{1,2\}$ let r_k^n be the length of the k-th longest orthogonal axis of the ellipsoid $J_n N$, where N is the unit circle with the center v_0. It means that the value r_k^n measures the expansion or contraction in the neighborhood of the orbit starting at v_0 during n first iterations. If the following limit exists $L_k = \lim_{n\to\infty} (r_k^n)^{1/n}$ it is called the k-th Lyapunov number and moreover k-th Lyapunov exponent of v_0 is $h_k = \ln L_k$. If $L_k > 1$ then $h_k > 0$, which means that two initially close trajectories can move away to each other. On other side if $0 < L_k < 1$ then $h_k < 0$, which means that two initially close trajectories can stay close to each other.

The particular algorithm for computing Lyapunov exponents uses an indirect approach. It is based on Wolf's ideas given in [1, 13] and can be briefly described as follows: (i) we start with initial orthonormal basis $\{w_1^0, w_2^0\}$ of the space R^2, that sufficiently characterize the initial circle N and further compute the vectors $z_1 = Df(v_0)w_1^0$ and $z_1 = Df(v_0)w_1^0$, (ii) use vectors $\{z_1, z_2\}$ and Gramm-Schmidt orthogonalization method to find orthogonal basis $\{y_1^1, y_2^1\}$, (iii) set $w_1^1 = y_1^1, w_2^1 = y_2^1$, (iii) repeat steps (i), (ii) and (iii) for sufficiently large number of steps n, (iv) the good approximation for total expansion r_k^n where $k \in \{1,2\}$ is $\|w_k^n\|^{1/n}$, where $\|.\|$ is the Euclidean norm at R^2.

Unfortunately the given algorithm is not a good one for a particular numerical computation. To avoid the computation with large and small numbers it is possible to notice that $r_k^n \approx \|y_k^n\| \cdot \ldots \cdot \|y_k^1\|$. If we take the logarithm of the latter formula we can summarize that the formula $n^{-1} \cdot \sum_{i=1}^{n} \ln y_k^i$ provide a good approximation of k-th largest Lyapunov exponent. The described algorithm for map (14) was implemented in Matlab. We have found that Lyapunov exponents for initial state $v_0 = (0.1, 0.1)$, parameter $\lambda = 6.25$ and parameter $\varepsilon = 2.2 \cdot 10^{-16}$ can be approximated by values $h_1 \approx 0.1616$ and $h_2 \approx 0.1605$ respectively. Since at least one value of Lyapunov exponent is positive we can conclude that for the given value of parameter of λ the map (14) is sensitive to initial conditions. It means that it is possible to consider that the given map shows features typical for deterministic chaos.

4 Conclusion

The paper introduced a nonlinear version of Cournot duopoly model. The essential assumption of the model is a nonlinear inverse demand function. In particular and similarly as in [9] the assumption that the quantity demanded is reciprocal to price was used. The equilibrium was found and conditions of its stability were established. Provided there is a sufficiently large proportion between costs of two firms in duopoly game it was shown that there exist nonlinear phenomena such as bifurcation of limit values of production or deterministic chaos. To prove a sensitive dependence on initial condition, that accompanies deterministic chaos, the concept of Lypunov exponent has been used.

In our future work we would like to improve and test the algorithm for computation of Lyapunov exponents. Its implementation in Matlab also deserves more tests and improvements.

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