Marques and Trojovský *Advances in Difference Equations* (2019) 2019:392
https://doi.org/10.1186/s13662-019-2327-6

Advances in Difference Equations
a SpringerOpen Journal

RESEARCH Open Access

On characteristic polynomial of higher order generalized Jacobsthal numbers

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Abstract
In this paper, we study a higher order generalization of the Jacobsthal sequence, namely, the (k,c)-Jacobsthal sequence $(J_n^{(k,c)})$ for any integers $n, k \geq 2$ and a real number $c > 0$. In particular, we find information about roots of its characteristic polynomial. For that purpose, we combine some powerful tools such as Marden's method, the Perron–Frobenius theorem, and the Eneström–Kakeya theorem.

MSC: 11Bxx; 11B83

Keywords: Linear recurrence sequence; Jacobsthal sequence; Generalized Jacobsthal sequence; Computation; Polynomial; Matrix theory; Digraph

1 Introduction

A sequence $(u_n)_n$ is a *homogeneous linear recurrence sequence* with coefficients $c_0, c_1, \ldots, c_{s-1}$, $c_0 \neq 0$, if

$$u_{n+s} = c_{s-1}u_{n+s-1} + \cdots + c_1 u_{n+1} + c_0 u_n, \tag{1}$$

for all non-negative integers n. A recurrence sequence is therefore completely determined by the *initial values* $u_0, u_1, \ldots, u_{s-1}$, and by the coefficients $c_0, c_1, \ldots, c_{s-1}$. The integer s is called the *order* of the linear recurrence. The *characteristic polynomial* of the sequence $(u_n)_{n\geq 0}$ is given by

$$\psi(x) = x^s - c_{s-1}x^{s-1} - \cdots - c_1 x - c_0 = (x - \alpha_1)^{m_1} \cdots (x - \alpha_\ell)^{m_\ell},$$

where the α_j's (which are distinct) are named the *roots* of the recurrence. Also, the recurrence $(u_n)_n$ has a *dominant root* if one of its roots has strictly largest absolute value. A fundamental result in the theory of recurrence sequences asserts that there exist uniquely determined non-zero polynomials $g_1, \ldots, g_\ell \in \mathbb{Q}(\{\alpha_j\}_{j=1}^{\ell})[x]$, with $\deg g_j \leq m_j - 1$ (m_j is the multiplicity of α_j as zero of $\psi(x)$), for $j = 1, 2, \ldots, \ell$, such that

$$u_n = g_1(n)\alpha_1^n + g_2(n)\alpha_2^n + \cdots + g_\ell(n)\alpha_\ell^n, \quad \text{for all } n. \tag{2}$$

For more details, see [11, Theorem C.1].

Let P, Q be non-zero integers and let $P^2 - 4Q \neq 0$. The sequences $(U_n(P,Q))_{n\geq 0}$ given for $n \geq 0$ by

$$U_{n+2}(P,Q) = P \cdot U_{n+1}(P,Q) - Q \cdot U_n(P,Q),$$

where $U_0(P,Q) = 0$, $U_1(P,Q) = 1$, is called the first Lucas sequence. For instance, if $P = 1$ and $Q = -1$, then $(U_n(1,-1))_{n\geq 0} = (F_n)_{n\geq 0}$ is the well-known *Fibonacci sequence*. The Fibonacci numbers are known for their amazing properties (see [9] for the history, properties, and applications of the Fibonacci sequence and some of its generalizations). When $P = 1$ and $Q = -2$, we find that $(U_n(1,-2))_{n\geq 0} = (J_n)_{n\geq 0}$ is the *Jacobsthal sequence*, which has many interesting properties (see [5]). An explicit formula for J_n is

$$J_n = \frac{2^n - (-1)^n}{3}.$$

There are several generalizations for Fibonacci numbers. For example, let $k \geq 2$ and denote $F^{(k)} := (F_n^{(k)})_{n\geq -(k-2)}$, the *k-generalized Fibonacci sequence* whose terms satisfy the recurrence relation

$$F_n^{(k)} = F_{n-1}^{(k)} + F_{n-2}^{(k)} + \cdots + F_{n-k}^{(k)}, \quad \text{for } n \geq 2, \tag{3}$$

with the initial conditions $F_{-(k-2)}^{(k)} = F_{-(k-3)}^{(k)} = \cdots = F_0^{(k)} = 0$ and $F_1^{(k)} = 1$.

The study of the behavior of the roots of the characteristic polynomial of a recurrence (which gives information about the asymptotic behavior of the sequence) has a very long history and it became more popular after the seminal work of Baker on effective lower bounds for linear forms in logarithms. For example, as a consequence of Baker's theory (see [1]) we have: *Let (u_n) be a recurrence sequence. Suppose that (u_n) is a sequence of integers of the form*

$$u_n = a\alpha^n + O\big(|\alpha|^{\theta n}\big), \quad \textit{with } \theta \in (0,1),$$

where a and α are non-zero algebraic numbers, with $|\alpha| > 1$ and such that $u_n - a\alpha^n \neq 0$ for all n. Then the equation

$$u_n = y^p, \quad u_n \notin \{0, \pm 1\},$$

implies $p < C$, where $C > 0$ is an effective constant, which depends only on the parameters of the recurrence (u_n). This result can be applied to k-generalized Fibonacci numbers. In fact, since it is known that the characteristic polynomial of $(F_n^{(k)})_n$, namely,

$$\psi_k(x) := x^k - x^{k-1} - \cdots - x - 1,$$

has just one zero outside the unit circle and all the zeros are simple (as can be found in [6]), we have $F_n^{(k)} = a\alpha^n + O(1)$. Actually, we remark that the case $k = 2$ was solved completely in 2003, by Bugeaud *et al* [3, Theorem 1].

Papers [8] studied some generalized sequences of (3) and authors proved similar properties of roots of the characteristic polynomials.

Here, we are interested in the following generalization of the Jacobsthal sequence.

Table 1 Examples of (k,c)-Jacobsthal sequences for some integer values of k and c

(k,c)	First 10 non-zero terms of $J_n^{(k,c)}$
(2, 2)	1, 1, 3, 5, 11, 21, 43, 85, 171, 341
(3, 5)	1, 1, 2, 8, 15, 33, 88, 196, 449, 1085
(4, 7)	1, 1, 2, 4, 14, 27, 59, 128, 312, 688

Definition 1 Let $k \geq 2$ be an integer and let $c > 0$ be a real number. The (k,c)-Jacobsthal sequence $(J_n^{(k,c)})_{n\geq -(k-2)}$ is defined by the recurrence

$$J_n^{(k,c)} = J_{n-1}^{(k,c)} + \cdots + J_{n-k+1}^{(k,c)} + c \cdot J_{n-k}^{(k,c)}, \quad \text{for } n \geq 2,$$

with the initial values $J_{-(k-2)}^{(k,c)} = J_{-(k-3)}^{(k,c)} = \cdots = J_0^{(k,c)} = 0$ and $J_1^{(k,c)} = 1$.

Some special cases of (k,c)-Jacobsthal sequences are listed in Table 1. Let $f_{k,c}(x)$ be the characteristic polynomial of the (k,c)-Jacobsthal sequence. Our first results are related to the zeros of $f_{k,c}(x)$. More precisely, we proved the following theorems.

Theorem 1 *Let $k \geq 2$ be an integer and let $c > 0$ be a real number. Set $f_{k,c}(x) = x^k - x^{k-1} - \cdots - x - c$, then $f_{k,c}(x)$ has a simple dominant zero and, moreover*:

(i) *$f_{k,2}(x)$ has 2 as the dominant zero and all the other zeros lie on the boundary of unit circle.*
(ii) *If $c > 2$, then $f_{k,c}(x)$ has a dominant zero $\alpha > 2$ and all the other zeros lie outside the closed unit circle.*
(iii) *If $c \in (0,2)$, then $f_{k,c}(x)$ has a dominant zero $\alpha \in (1,2)$ and all the other zeros lie inside the unit circle.*

Theorem 2 *For $k \geq 2$, all the zeros of $f_{k,c}(x)$ are simple if some of the items below is true*

(i) $c \in (1,2]$;
(ii) *$c > 2$ and $k \geq \sqrt{\frac{8c(c-1)}{(c-2)^2}} + \frac{3c-2}{c-2}$.*

Moreover, there are at most two zeros α_+ and α_- with multiplicity greater than one and they must have the form

$$\alpha_\pm = \frac{3ck + 2 - c - 2k \pm \sqrt{(c + 2k - 3ck - 2)^2 - 8c(c-1)k^2}}{2(c-1)k}.$$

As a consequence of the previous theorem, we find that the following result holds for integer higher order Jacobsthal recurrences.

Corollary 1 *If c and k are positive integers, then all the zeros of $f_{k,c}(x)$ are simple.*

2 Auxiliary results

In this section, we shall present some results which will be essential ingredients in the proof of our results.

Our first tool is related to a method of Marden [10, Chapter X] for calculating the number of zeros of a polynomial within the unit circle. Here we shall state only a particular case which is convenient to us. For that, first, consider a polynomial $f(x) = a_0 + a_1x + \cdots + a_nx^n$

with real coefficients and denote by $f^*(x)$ its *reciprocal polynomial*, i.e., $f^*(x) = x^n f(1/x)$. We define the *Schur transform* of $f(x)$, denoted by $\mathrm{Tf}(x)$, by

$$\mathrm{Tf}(x) = a_0 f(x) - a_n f^*(x).$$

A particular case of Marden's result [10, Lemma 42.1] is the following.

Theorem 3 *Let $f(x) = a_0 + a_1 x + \cdots + a_n x^n$ be a polynomial with real coefficients. If $\delta(f) := a_0^2 - a_n^2 \neq 0$, then $f(x)$ and $\mathrm{Tf}(x)$ have the same number of zeros on the boundary of the unit circle.*

Another useful and very important result is due to Eneström and Kakeya [4, 7].

Theorem 4 *Let $f(x) = a_0 + a_1 x + \cdots + a_n x^n$ be an n-degree polynomial with real coefficients. If $0 \leq a_0 \leq a_1 \leq \cdots \leq a_n$, then all zeros of $f(x)$ lie in $|x| \leq 1$.*

Further, we shall use the Perron–Frobenius theorem from the linear algebra.

Theorem 5 *Let A be a square matrix with non-negative real entries. If A^k is a positive matrix (i.e., a matrix having all positive entries), then A has a positive eigenvalue of multiplicity* 1 *and strictly greater in absolute value than all other eigenvalues.*

We still shall use two well-known trigonometric formulas: For $\alpha \neq 2k\pi$, where k is any integer, we have

$$\sin(\phi) + \sin(\phi + \alpha) + \cdots + \sin(\phi + n\alpha) = \frac{\sin\frac{(n+1)\alpha}{2}\sin(\phi + \frac{n\alpha}{2})}{\sin\frac{\alpha}{2}} \tag{4}$$

and

$$\cos(\phi) + \cos(\phi + \alpha) + \cdots + \cos(\phi + n\alpha) = \frac{\sin\frac{(n+1)\alpha}{2}\cos(\phi + \frac{n\alpha}{2})}{\sin\frac{\alpha}{2}}. \tag{5}$$

With these tools at hand we are ready to deal with the proof of our results.

3 Proof of Theorem 1

For proving that $f_{k,c}(x)$ has a dominant zero for all values of $c > 0$, we shall use the connection between recurrence sequences and linear algebra (eigenvalues, characteristic polynomial etc.). First, we note that $f_{k,c}(x)$ is the characteristic polynomial of its companion $k \times k$ matrix

$$A_c = \begin{bmatrix} 0 & 0 & \cdots & 0 & c \\ 1 & 0 & \cdots & 0 & 1 \\ 0 & 1 & \cdots & 0 & 1 \\ \vdots & \vdots & \ddots & \vdots & \\ 0 & 0 & \cdots & 1 & 1 \end{bmatrix}$$

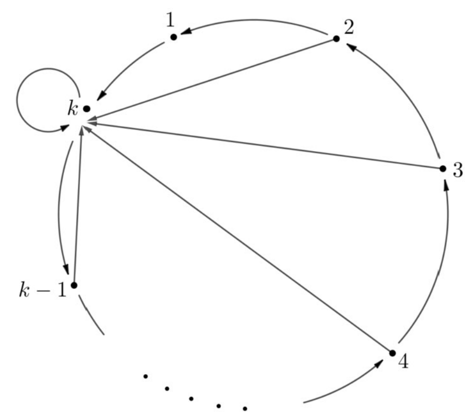

Figure 1 The digraph corresponding to the companion matrix A_c

That is, $f_{k,c}(x) = \det(xI - A_c)$ and then the roots of the recurrence of the (k,c)-Jacobsthal sequence are precisely the eigenvalues of the matrix A_c. Thus, in order to prove the existence of a dominant zero for $f_{k,c}(x)$, it suffices to prove the existence of an eigenvalue with absolute value strictly greater than the absolute value of all other eigenvalues (this largest absolute value is called *spectral radius*). For proving that, we shall use Theorem 5. So, we must prove that A_c^n is a positive matrix, for some $n \geq 1$. In fact, we claim that A_c^k is a positive matrix. To prove this assertion, we shall use a fact from graph theory, which says that, for $A_c = (a_{i,j})$, we can look at the directed hypergraph of k vertices where arcs correspond to positive entries of A_c (including a loop $k \to k$). So, the entry (i,j) of A_c^k is positive if you can get from i to j in k steps (loops are allowed). Since $a_{i,k} > 0$, for all $1 \leq i \leq k$ and $a_{i+1,i} > 0$, for all $1 \leq i \leq k-1$, our graph is like Fig. 1 (we refer the reader to [2, p. 78] for this results and other facts about combinatorial matrix theory). So, the first step is to go from i to k (since $a_{i,k} > 0$), after we stay in the loop $k \to k$ for $j-1$ steps and then we go to j in $k-j$ steps. So, we reach j from i in $1 + (j-1) + (k-j) = k$ steps. Since the pair (i,j) is arbitrary, the matrix A_c^k has only positive entries and by the Perron–Frobenius theorem, we find that $f_{k,c}(x)$ has a simple dominant zero, as desired.

In order to prove the items, we observe that, for $c = 1$, the $(k,1)$-Jacobsthal sequence is exactly the k-generalized Fibonacci sequence (and the result is already known for this last sequence). So, we may suppose, in all that follows, that $c \neq 1$.

Proof of (*i*) The proof follows directly from the fact that

$$f_{k,2}(x) = (x-2)\frac{x^k - 1}{x - 1}.$$

□

Proof of (*ii*) Note that by Descartes' sign rule, $f_{k,c}(x)$ has only one positive zero, say α. Also, $f_{k,c}(2) = 2 - c < 0$ and then $\alpha > 2$, by the Intermediate Value Theorem, together with the fact that $f_{k,c}(x)$ tends to infinity as $x \to \infty$.

Define $g_c(x) = -f_{k,c}^*(1/x) = cx^k + x^{k-1} + \cdots + x - 1$. Thus $\beta := 1/\alpha \in (0, 1/2)$ is a zero of $g_c(x)$. Note that

$$g_c(x) = (x-\beta)\big(cx^{k-1} + (c\beta+1)x^{k-2} + \cdots + \big(c\beta^{k-1} + \beta^{k-2} + \cdots + \beta + 1\big)\big).$$

Write $\psi_c(x) := g_c(x)/(x-\beta)$. Now, we shall prove that the coefficients of $\psi_c(x)$ are in the decreasing order. In fact, since $c(1-\beta) > 1$ (because $\beta < 1/2$ and $c > 2$), we have $c > c\beta + 1$.

So, by the same reason, we have

$$c\beta^{j-1} + \sum_{i=0}^{j-2} \beta^i > c\beta^j + \sum_{i=0}^{j-1} \beta^i,$$

for all $j \in [2,k]$ (since the above inequality is equivalent to $c > c\beta + 1$). Therefore, by the Eneström–Kakeya theorem, all the zeros of $\psi_c(x)$ satisfy $|x| \le 1$ and so, all the zeros of $f_{k,c}(x)$ satisfy $|x| \ge 1$. Now, it suffices to prove that these zeros do not lie on the unit circle. For that, we shall use Marden's method. Since $\delta(f_{k,c}) = (-c^2) - 1^2 \neq 0$, then, by Marden's theorem, the polynomials $f_{k,c}(x)$ and $\mathrm{Tf}_{k,c}(x) = (-c)^2 f_{k,c}(x) - x^k f_{k,c}(1/x)$ have the same number of zeros on the boundary of the unit circle. After some calculations, we obtain

$$\mathrm{Tf}_{k,c}(x) = x^{k-1} + x^{k-2} + \cdots + x + c - 1.$$

Let us prove that this polynomial does not have a zero with absolute value equal to 1. Indeed, suppose that $x = \exp(i\theta) = \cos(\theta) + i\sin(\theta)$ (for some $\theta \in (0,\pi)$, since -1 and 1 are not zeros of $\mathrm{Tf}_{k,c}(x)$) satisfy $\mathrm{Tf}_{k,c}(x) = 0$. Thus, we use De Moivre's formula and by combining the real and imaginary parts, we obtain

$$\cos(\theta) + \cos(2\theta) + \cdots + \cos\bigl((k-1)\theta\bigr) = 1 - c$$

and

$$\sin(\theta) + \sin(2\theta) + \cdots + \sin\bigl((k-1)\theta\bigr) = 0.$$

By using Eqs. (4) and (5), we arrive at

$$\frac{\sin((k-1)\theta/2)\cos(k\theta/2)}{\sin(\theta/2)} = 1 - c \tag{6}$$

and

$$\frac{\sin((k-1)\theta/2)\sin(k\theta/2)}{\sin(\theta/2)} = 0. \tag{7}$$

Since $c \neq 1$, from (7), one has $\sin(k\theta/2) = 0$ and so $k\theta/2 = \ell\pi$, for some integer ℓ. Thus, $\cos(k\theta/2) = (-1)^\ell$ and we use the sine addition formula to get $\sin((k-1)\theta/2) = (-1)^{\ell+1}\sin(\theta/2)$. We combine this fact with (6) to arrive at the absurdity that $-1 = 1 - c$, i.e., $c = 2$. So, all the zeros of $f_{k,c}(x)$ satisfy $|x| > 1$. □

Proof of (*iii*) Again, by Descartes' sign rule, $f_{k,c}(x)$ has only one positive zero, say α. Also, $f_{k,c}(2) = 2 - c > 0$ and $f_{k,c}(1) = 2 - k - c < 0$ and then $\alpha \in (1,2)$, by the intermediate value theorem.

Define $g_c(x) = -f^*_{k,c}(1/x) = cx^k + x^{k-1} + \cdots + x - 1$. Thus $\beta := 1/\alpha > 1/2$ is a zero of $g_c(x)$. Note that

$$g_c(x) = (x - \beta)\bigl(cx^{k-1} + (c\beta + 1)x^{k-2} + \cdots + \bigl(c\beta^{k-1} + \beta^{k-2} + \cdots + \beta + 1\bigr)\bigr).$$

Write $\psi_c(x) := g_c(x)/(x-\beta)$. Then

$$\psi_c^*(x) = \left(c\beta^{k-1} + \beta^{k-2} + \cdots + \beta + 1\right)x^{k-1} + \cdots + (c\beta + 1)x + c.$$

Now the coefficients of the previous polynomial are in increasing order. In fact, as in the proof of item (ii), it is enough to prove that $c\beta + 1 \geq c$. This holds, because $c(1-\beta) < 2 \cdot (1 - 1/2) = 1$ (since $c < 2$ and $\beta > 1/2$). Thus, we use the Eneström–Kakeya theorem to ensure that all the zeros of $\psi_c^*(x)$ satisfy $|x| \leq 1$, so, all the zeros of $g_c(x)$ satisfy $|x| \geq 1$ and finally all the zeros of $f_{k,c}(x)$ (different of α) satisfy $|x| \leq 1$. Now, the proof that there is no zero of $f_{k,c}(x)$ on the boundary of the unit circle is the same as in the previous item (since in that proof the absurdity was that $c = 2$). This completes the proof. □

4 Proofs of Theorem 2 and Corollary 1

4.1 Proof of Theorem 2

Let $g_c(x) = (x-1)f_{k,c}(x) = x^{k+1} - 2x^k - (c-1)x + c$. By Descartes' sign rule this polynomial has two positive real zeros counting multiplicity. One of them is $x = 1$ and the other, which is a zero of $f_{k,c}(x)$, must be simple (note that $f_{k,c}(1) = 2 - k - c < 0$). For the same reason (Descartes' sign rule for $g_c(-x)$), we find that $g_c(x)$ has exactly one negative zero when k is even and exactly two negative zeros or none negative zeros when k is odd. So, in the even case the real zeros must be simple.

In conclusion, a possible zero α of $g_c(x)$ with multiplicity greater than one must be a non-real number. Thus, since $g_c(\alpha) = 0$ and $g_c'(\alpha) = 0$ have to hold, we can combine these equalities to obtain

$$0 = h_c(\alpha) := \alpha g_c'(\alpha) - (k+1)g_c(\alpha) = 2\alpha^k + (c-1)k\alpha - c(k+1).$$

Also,

$$0 = (\alpha - 2)h_c(\alpha) - 2g_c(\alpha) = (c-1)k\alpha^2 + (c + 2k - 3ck - 2)\alpha + 2ck.$$

This implies that we have two possibilities for α, namely,

$$\alpha_{\pm} = \frac{3ck + 2 - c - 2k \pm \sqrt{(c + 2k - 3ck - 2)^2 - 8c(c-1)k^2}}{2(c-1)k}.$$

However, for any c and k as in items (i) or (ii), we obtain

$$(c + 2k - 3ck - 2)^2 \geq 8c(c-1)k^2, \tag{8}$$

showing that α is real. Inequality (8) we can rewrite as

$$(c-2)^2k^2 - 2(c-2)(3c-2)k + (c-2)^2 \geq 0 \tag{9}$$

or

$$(c-2)^2\left(k^2 + 1\right) \geq -2(2-c)(3c-2)k. \tag{10}$$

For $c \in (1,2]$ we can see from (10) that (8) surely holds for any non-negative k. Let us consider $c \in (0,1] \cup (2,\infty)$. The discriminant D of the quadratic polynomial (in variable k) on the left-hand side of inequality (9) is equal to $32c(c-1)(c-2)^2$. Clearly, $D \leq 0$ for $c \in (0,1]$, hence (8) holds for any k. Similarly, $D > 0$ for $c > 1$, thus we have to solve only the case $c > 2$. Zeros of the quadratic polynomial from the left-hand side of inequality (9) are

$$k_{\pm} = \frac{3c-2}{c-2} \pm 2\sqrt{2}\sqrt{\frac{c(c-1)}{(c-2)^2}}. \tag{11}$$

These zeros are dependent on the parameter c, hence we will define two functions k_+ and k_-, $k_{\pm} : (2,\infty) \to \mathbb{R}$, given by (11). We get easily that $\lim_{c\to 2^+} k_-(c) = 0$, $\lim_{c\to 2^+} k_+(c) = +\infty$ and $\lim_{c\to\infty} k_{\pm}(c) = 3 \pm 2\sqrt{2}$. The derivatives of these functions are

$$k'_{\pm}(c) = \frac{-4\sqrt{c(c-1)} \pm \sqrt{2}(2-3c)}{(c-2)^2\sqrt{c(c-1)}}.$$

We easily see that the function $k_+(c)$ is decreasing and the function $k_-(c)$ is increasing in $(2,\infty)$. Thus, for $c > 2$ the function k_- is bounded, concretely $0 < k_-(c) < 3 - 2\sqrt{2} < 0.2$ and function k_- is bounded from below by $3 + 2\sqrt{2} > 2$. Hence for $c > 2$ we have the following condition for k:

$$k \geq \frac{3c-2}{c-2} + \sqrt{\frac{8c(c-1)}{(c-2)^2}}.$$

This completes the proof.

4.2 Proof of Corollary 1

First, let us suppose that $k > 5$. We will again use the function k_+, defined above, and its properties. If c is an integer, then the maximum of $k_+(c)$ happens when $c = 3$ and this maximum is equal to $k_+(3) = 7 + 4\sqrt{3} < 14$. So, if $k \geq 14$, then, by Theorem 2, $f_{k,c}(x)$ has only single zeros and this fact does not depend on c. However, if c increases, function $k_+(c)$ decreases and then we can obtain better lower bounds for k which can lead, by computational methods, to our desired result. For evaluating this task, we shall define some commands in *Wolfram Mathematica*. First, the following function $r(c)$:

```
r[c_] := IntegerPart[(-2 + 3 c)/(-2 + c) +
        2 Sqrt[2] Sqrt[(-c + c^2)/(-2 + c)^2]]
```

The functions $f_{k,c}(x)$ (here $F(x,k,c)$):

```
F[x_, k_, c_] := x^k - Sum[x^j, {j, 1, k - 1}] - c
```

Its derivative (in relation to x):

```
G[x_, k_] := k*x^(k -1) - Sum[j*x^(j - 1), {j, 1, k - 1}]
```

We find that the possible multiple zeros for $f_{k,c}(x)$ happen for $5 < k \leq r(c)$. However, $r(c) = 5$, for all $c \geq 51$ and so there are none of these kind of zeros for these cases. So, we can consider only the case $3 \leq c \leq 50$. We know that α is a multiple zero of $f_{k,c}(x)$ if $f_{k,c}(\alpha) = f'_{k,c}(\alpha) = 0$. The next function will calculate the maximum between the number of solutions for a fixed c and k in the range $6 \leq k \leq r(c)$:

```
s[c_] := Max[
 Table[ Length[NSolve[F[x, k, c] == 0 && G[x, k] == 0, x ]],
   {k, 6, h[c]} ]]
```

Finally, we make a table for c from 3 to 50 by the following input:

```
Table[ s[c], {c, 3, 50} ]
```

The output is

```
{0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,
0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,
 0, 0, 0, 0, 0, 0, 0, 0}
```

This means that there are no multiple zeros if $k > 5$.

For the cases, $k = 2, 3, 4$ or 5, we can use *Mathematica* to return the exact zeros of $f'_{k,c}(x)$ (for $k \leq 5$, we have a closed formula for these zeros, which does not depend on c). After that, we consider the linear equation $f_{k,c}(y) = 0$ (in the variable c) for y being a root of $f'_{k,c}(x) = 0$. Then we again use *Mathematica* to see that none of the returned values of c is a positive integer. This concludes the proof.

Acknowledgements
The authors thank the anonymous referees for their careful corrections and very helpful and detailed comments, which have significantly improved the presentation of this paper.

Funding
This research was funded by the Project of Excelence PřF UHK 01/2018, University of Hradec Králové, Czech Republic.

Availability of data and materials
The data used to support the findings of this paper are included within the article. Data sharing not applicable to this article as no datasets were generated or analysed during the current study.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
Authors worked together to produce the results presented here. All authors read and approved the final manuscript.

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Received: 3 June 2019 Accepted: 4 September 2019 Published online: 11 September 2019

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