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On the Sum of Reciprocal of Polynomial Applied to Higher Order Recurrences

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Abstract: Recently a lot of papers have been devoted to partial infinite reciprocal sums of a higher-order linear recursive sequence. In this paper, we continue this program by finding a sequence which is asymptotically equivalent to partial infinite sums, including a reciprocal of polynomial applied to linear higher order recurrences.

Keywords: linear recurrence; higher order sequence; Landau symbol; asymptotic equivalence

MSC: 11B39; 11C08

1. Introduction

A sequence $(u_n)_n$ is a linear recurrence sequence with coefficients $c_0, c_1, \ldots, c_{s-1}$, where $c_0 \neq 0$ if:

$$u_{n+s} = c_{s-1}u_{n+s-1} + \cdots + c_1 u_{n+1} + c_0 u_n, \tag{1}$$

for all non-negative integers n. A recurrence sequence is therefore completely determined by the initial values $u_0, \ldots, u_{s-1} \in \mathbb{C}$, and by the coefficients $c_0, c_1, \ldots, c_{s-1} \in \mathbb{C}$. The integer s is called the 'order' of the linear recurrence. The characteristic polynomial of the sequence $(u_n)_{n\geq 0}$ is given by:

$$\psi(x) = x^s - c_{s-1}x^{s-1} - \cdots - c_1 x - c_0 = (x-\alpha_1)^{m_1} \cdots (x-\alpha_\ell)^{m_\ell},$$

where the α_j (which are distinct) are called the 'roots' of the recurrence. Moreover, the recurrence $(u_n)_n$ has a 'dominant root' if one of its roots has strictly largest absolute value. A fundamental result in the theory of recurrence sequences asserts that there exist uniquely determined non-zero polynomials $g_1, \ldots, g_\ell \in \mathbb{Q}(\{\alpha_j\}_{j=1}^{\ell})[x]$, with $\deg g_j \leq m_j - 1$ (m_j is the multiplicity of α_j as root of $\psi(x)$), for $j = 1, \ldots, \ell$, such that:

$$u_n = g_1(n)\alpha_1^n + \cdots + g_\ell(n)\alpha_\ell^n, \text{ for all } n. \tag{2}$$

For more details, see [1] (Theorem C.1). The great importance of linear recurrence relations in practice can be seen, for example, from recently published papers [2–5].

The Lucas sequence $(U_n)_{n\geq 0}$ given by $U_{n+2} = aU_{n+1} + bU_n$, for $n \geq 0$, where $U_0 = 0$, $U_1 = 1$ and the values a and b are previously fixed, is an example of a linear recurrence of order 2 (also called 'binary'). For instance, if $a = b = 1$, then $(U_n)_{n\geq 0} = (F_n)_{n\geq 0}$ is the well-known Fibonacci sequence. The Fibonacci numbers are known for their amazing properties (see [6] for the history, properties and rich applications of the Fibonacci sequence and some of its generalizations).

In 2008, Ohtsuka and Nakamura [7] studied the partial infinite sums of reciprocal Fibonacci numbers and proved that:

$$\left\lfloor \left(\sum_{k=n}^{\infty} \frac{1}{F_k} \right)^{-1} \right\rfloor = \begin{cases} F_{n-2}, & \text{if } n \equiv 0 \pmod{2},\ n \geq 2; \\ F_{n-2} - 1, & \text{if } n \equiv 1 \pmod{2},\ n \geq 1. \end{cases}$$

When $a = 2$ and $b = 1$, we have another important sequence, namely, the sequence of the *Pell numbers* $(P_n)_n$. In 2012, Wenpeng and Tingting [8] proved that:

$$\left\lfloor \left(\sum_{k=n}^{\infty} \frac{1}{P_k} \right)^{-1} \right\rfloor = \begin{cases} P_n + P_{n-2}, & \text{if } n \equiv 0 \pmod{2},\ n \geq 2; \\ P_{n-1} + P_{n-2} - 1, & \text{if } n \equiv 1 \pmod{2},\ n \geq 1. \end{cases}$$

In 2011, Holliday and Komatsu [9] obtained the infinite sum of the reciprocal of the generalized Fibonacci sequence $\{U_n\}$, when $a = p$ and $b = 1$

$$\left\lfloor \left(\sum_{k=n}^{\infty} \frac{1}{U_k} \right)^{-1} \right\rfloor = \begin{cases} U_n - U_{n-1}, & \text{if } n \equiv 0 \pmod{2},\ n \geq 2; \\ U_{n-1} - U_{n-1} - 1, & \text{if } n \equiv 1 \pmod{2},\ n \geq 1. \end{cases}$$

In 2016, Basbük and Yazlik [10] studied generalized bi-periodic Fibonacci numbers, defined by Edson and Yayenie [11] in the following manner:

$$q_n = \begin{cases} aq_{n-1} + q_{n-2}, & \text{if } n \equiv 0 \pmod{2},\ n \geq 2; \\ bq_{n-1} + q_{n-2}, & \text{if } n \equiv 1 \pmod{2},\ n \geq 2, \end{cases}$$

with $q_0 = 0$ and $q_1 = 1$. They derived the following partial infinite sum of reciprocal for this sequence:

$$\left\lfloor \left(\sum_{k=n}^{\infty} \left(\frac{a}{b}\right)^{\Psi(k)} \frac{1}{q_k} \right)^{-1} \right\rfloor = \begin{cases} q_n - q_{n-1}, & \text{if } n \equiv 0 \pmod{2},\ n \geq 2; \\ q_{n-1} - q_{n-1} - 1, & \text{if } n \equiv 1 \pmod{2},\ n \geq 1, \end{cases}$$

where $\Psi(k) = s(k+1) - s(n+1) - (-1)^n \lfloor \frac{k-n}{2} \rfloor$, with:

$$s(n) = \begin{cases} 0, & \text{if } n \equiv 0 \pmod{2}; \\ 1, & \text{if } n \equiv 1 \pmod{2}. \end{cases}$$

Choi and Choo [12] derived the formulas for the following sums of reciprocals of products of Fibonacci and Lucas numbers:

$$\left\lfloor \left(\sum_{k=n}^{\infty} \frac{1}{F_k L_{k+m}} \right)^{-1} \right\rfloor \text{ and } \left\lfloor \left(\sum_{k=n}^{\infty} \frac{1}{L_k F_{k+m}} \right)^{-1} \right\rfloor.$$

Choi and Choo [13] proceeded in study of sequence from [10] and found a formula for the following sum of reciprocals of the squares of generalized Fibonacci numbers:

$$\left\lfloor \left(\sum_{k=n}^{\infty} \left(\frac{a}{b}\right)^{1-\Psi(k)} \frac{1}{q_k^2} \right)^{-1} \right\rfloor.$$

We point out to the reader the series of papers [14–21], where the authors deal, in particular, with partial sums related to the omnipresent Riemann zeta function $\zeta(s)$. For instance, Xin [17] proved that:

$$\left\lfloor \left(\sum_{k=n}^{\infty} \frac{1}{k^2} \right)^{-1} \right\rfloor = n - 1,$$

where n is any positive integer.

In 2013, Kiliç and Arikan [22] studied a problem which differs slightly from the one in [7], namely, they determined the nearest integer to $(\sum_{k=n}^{\infty}(1/u_k))^{-1}$. Specifically, suppose that $\|x\| = \lfloor x + 1/2 \rfloor$ (the nearest integer formula) and let $(u_n)_n$ be an integer sequence satisfying the recurrence formula:

$$u_n = pu_{n-1} + qu_{n-2} + u_{n-3} + \cdots + u_{n-k},$$

for any positive integer $p \geq q$ and $n \geq k$. Then, they proved the existence of a positive integer n_0 such that:

$$\left\| \left(\sum_{k=n}^{\infty} \frac{1}{u_k} \right)^{-1} \right\| = u_n - u_{n-1},$$

for all $n \geq n_0$.

There are many generalizations of this result (see, for example, [15] and references therein). In all these cases, the authors considered sequences whose characteristic polynomial has only one root outside the closed unit disc and all the other roots lie inside this disc. To ensure that, they considered some particular recurrences and proved that this property holds by using a calculus approach to their characteristic polynomial.

Here we are interested in a huge class of recurrences. For this reason, we shall look at the problem from another viewpoint. Specifically, we recall that two sequences $(u_n)_n$ and $(v_n)_n$ are called 'asymptotically equivalent' if u_n/v_n tends to 1 as $n \to \infty$ (we write $u_n \sim v_n$).

In this paper, we shall prove the following result:

Theorem 1. *Let $(u_n)_n$ be a linear recurrence having a simple dominant root. Let $P(z) \in \mathbb{C}[z]$ be a non-constant polynomial. Then, the sequences:*

$$\left(\left(\sum_{k=n}^{\infty} \frac{1}{P(u_k)} \right)^{-1} \right)_n \text{ and } (P(u_n) - P(u_{n-1}))_n$$

are asymptotically equivalent.

Moreover, in the final section, we shall make some computations of 'relative error of this asymptotic behavior', which will be related to some special forms of recurrences $(u_n)_n$. The calculations performed in this paper took several minutes using software Mathematica on a 2.5 GHz Intel Core i5 4GB Mac OSX.

2. Proof of the Theorem

In what follows, we shall use the usual notation O which denotes the 'big-oh' Landau symbol.

If $\alpha_1, \ldots, \alpha_\ell$ are the roots of $(u_n)_n$ with $|\alpha_1| > |\alpha_2| \geq |\alpha_3| \geq \cdots \geq |\alpha_\ell|$, we then write $\alpha := \alpha_1$ and $\beta := \alpha_2$. By hypothesis, α is a simple root of the function ψ and so the polynomial $g_1(n)$ is a constant, say a. By the formula in Equation (2), we have that:

$$u_k = a\alpha^k + O(k^s \beta^k).$$

We may suppose that $|\alpha| > 1$ (otherwise u_k tends to 0 as $k \to \infty$ and the result clearly holds). Further, we assume that $|\beta| \geq 1$ (if not, all the other roots of the recurrence would lie inside the open unit disc and the Kiliç-Arikan method could be used, since we can write $u_k = a\alpha^k + O(c^{-k})$, for some $c > 1$). Thus, we have:

$$u_k = a\alpha^k(1 + O((\beta/\alpha)^{k/2})),$$

where we used that $|\alpha/\beta|^{k/2} > k^s$ for all sufficiently large k. Suppose, without loss of generality, that $P(z) = z^m + \sum_{j=0}^{m-1} b_j z^j$. Since:

$$u_k^j = a^j\alpha^{kj}(1 + O((\beta/\alpha)^{k/2})),$$

we obtain:

$$P(u_k) = P(a\alpha^k)(1 + O((\beta/\alpha)^{k/2})).$$

Now, we shall use that $1/(1 \pm \epsilon) = 1 + O(\epsilon)$ (for all ϵ sufficiently small) to write:

$$\frac{1}{P(u_k)} = \frac{1}{P(a\alpha^k)}(1 + O((\beta/\alpha)^{k/2})).$$

Thus:

$$\sum_{k=n}^{\infty} \frac{1}{P(u_k)} = \sum_{k=n}^{\infty} \frac{1}{P(a\alpha^k)} + O\left(\alpha^{-nm}(\beta/\alpha)^{n/2}\right),$$

where we used that $|P(a\alpha^k)| > |a^m||\alpha^{km}|/2$, for all sufficiently large k (when $|\alpha| > 1$, for the case $|\alpha| = 1$, the estimate in the previous formula is still valid). In addition, we use that $P(a\alpha^k) = a^m\alpha^{km}(1 + O(\alpha^{-k}))$ to get:

$$\frac{1}{P(a\alpha^k)} = \frac{1}{a^m\alpha^{km}(1 + O(\alpha^{-k}))} = \frac{1}{a^m\alpha^{km}} + O(\alpha^{-k(m+1)}).$$

So, by using the geometric series, we arrive at:

$$\sum_{k=n}^{\infty} \frac{1}{P(a\alpha^k)} = \frac{\alpha^m}{a^m\alpha^{mn}(\alpha^m - 1)} + O(\alpha^{-n(m+1)}).$$

Therefore:

$$\sum_{k=n}^{\infty} \frac{1}{P(u_k)} = \frac{\alpha^m}{a^m\alpha^{mn}(\alpha^m - 1)}(1 + O((\beta/\alpha)^{n/2}))$$

and so:

$$\begin{aligned} \left(\sum_{k=n}^{\infty} \frac{1}{P(u_k)}\right)^{-1} &= \frac{a^m\alpha^{mn}(\alpha^m - 1)}{\alpha^m}(1 + O((\beta/\alpha)^{n/2})) \\ &= (a^m(\alpha^n)^m - a^m\alpha^{(n-1)m})(1 + O((\beta/\alpha)^{n/2})). \end{aligned} \quad (3)$$

This implies that, if $A_n := (\sum_{k=n}^{\infty} 1/P(u_k))^{-1}$, then:

$$A_n/(a^m(\alpha^n)^m - a^m(\alpha^{n-1})^m) \text{ tends to } 1 \text{ as } n \to \infty.$$

On the other hand:

$$\begin{aligned} P(u_k) &= P(a\alpha^k)(1 + O((\beta/\alpha)^{k/2})) \\ &= a^m\alpha^{km}(1 + O(\alpha^{-k}))(1 + O((\beta/\alpha)^{k/2})) \\ &= a^m\alpha^{km} + \alpha^{km}O(\nu^k), \\ &= a^m\alpha^{km} + \alpha^{km}O(\nu^{k-1}), \end{aligned}$$

where $\nu := \max\{|\beta/\alpha|^{1/2}, |\alpha|^{-1}\}$. Hence:

$$P(u_n) - P(u_{n-1}) = (a^m \alpha^{mn} - a^m \alpha^{m(n-1)})(1 + O(\nu^{n-1})) \tag{4}$$

which yields that:

$$\frac{P(u_n) - P(u_{n-1})}{a^m(\alpha^n)^m - a^m(\alpha^{n-1})^m} \quad \text{tends to 1,}$$

as $n \to \infty$. So, by combining the previous facts, we conclude that $A_n/(P(u_n) - P(u_{n-1}))$ tends to 1 as $n \to \infty$ and then $(A_n)_n \sim (P(u_n) - P(u_{n-1}))_n$ which completes the proof. □

3. Computational Aspects

We will now discuss 'relative error of an asymptotic behavior of our result in Theorem 1'. Using Identities (3) and (4) we get

$$\left| \frac{\left(\sum_{k=n}^{\infty} (P(u_k))^{-1} \right)^{-1}}{P(u_n) - P(u_{n-1})} - 1 \right| = \left| \frac{1 + O((\beta/\alpha)^{n/2})}{1 + O(\nu^{n-1})} - 1 \right| = O(\nu^{n-1}). \tag{5}$$

Example 1. *Determine the dominant root α and the second dominant root β as well as the magnitude of the relative error of asymptotic equivalence by Identity (5) for $n = 100$, for the following two sequences $(u_n)_n$ defined by linear recurrences of the fourth-order:*

$$\begin{aligned} u_n &= 2u_{n-1} + 7u_{n-2} + u_{n-3} + u_{n-4}, \\ u_n &= 2u_{n-1} + 7u_{n-2} + 5u_{n-3} + u_{n-4}. \end{aligned}$$

Requested computations are in Table 1. We used software Mathematica.

Table 1. Fourth order linear recurrences.

u_n	α	β	$\lvert\nu\rvert^{n-1}$
$u_n = 2u_{n-1} + 7u_{n-2} + u_{n-3} + u_{n-4}$	3.8851	1.7845	1.8847×10^{-17}
$u_n = 2u_{n-1} + 7u_{n-2} + 5u_{n-3} + u_{n-4}$	4.0489	−1	8.6442×10^{-31}

Example 2. *Let $(u_n)_n$ be the sequence defined by the fourth-order linear recurrence:*

$$u_n = u_{n-1} + 2u_{n-2} + u_{n-3} + u_{n-4},$$

with initial values $u_1 = u_2 = u_3 = 1$ and $u_4 = 2$. Let $P(z) = z^2 + z + 1$, $A_n := \sum_{k=n}^{\infty} 1/P(u_k)$ and $B_n := P(u_n) - P(u_{n-1})$. Examine the behavior of values A_n/B_n, the magnitude of the relative error of asymptotic equivalence given by Identity (5), for $n = 10, 100, 500$.

Using software Wolfram Mathematica, we can find that the two biggest roots (in the absolute value) of the characteristic polynomial of $(u_n)_n$ are $\alpha = 2.2026$ and $\beta = -1$. Table 2 shows the results.

Table 2. Values A_n/B_n the magnitude of the relative error of asymptotic equivalence.

n	A_n	B_n	A_n/B_n	$\lvert\nu\rvert^{n-1}$
10	≈66,125.65	65,940	1.0028	0.01916
100	$\approx 4.5 \times 10^{66}$	$\approx 4.5 \times 10^{66}$	$1 + 3 \times 10^{-34}$	6.6×10^{-18}
500	$\approx 2.9 \times 10^{341}$	$\approx 2.9 \times 10^{341}$	$1 + 10^{-100}$	1.3×10^{-86}

4. A Generalization for Higher Dimensional Recurrences

In this section, we shall point out what can be done for partial infinite reciprocal sums of higher dimensional recurrences $(\vec{u}_n)_n$, with $\vec{u}_n \in \mathbb{C}^d$, where d is a positive integer. As the proof is very similar to the proof of Theorem 1 in Section 2, we shall only remark on the main points of the proof in order to avoid many technical details. Suppose that we have the recurrence relation:

$$\vec{u}_{n+s} = c_{s-1}\vec{u}_{n+s-1} + \cdots + c_0\vec{u}_n,$$

with the initial values $\vec{u}_0, \ldots, \vec{u}_{s-1} \in \mathbb{C}^d$. We can write the sequence $(\vec{u}_n)_n$ in the following form:

$$(\vec{u}_n)_n = ((u_n^{(1)})_n, (u_n^{(2)})_n, \ldots, (u_n^{(d)})_n)\ ,$$

thus as a d-tuple of complex linear recurrences (or a $2d$-tuple by considering their real and imaginary parts). Note that each sequence $(u_n^{(i)})_n$, where $i \in \{1, 2, \ldots, d\}$, satisfies the same linear recurrence relation as the sequence $(\vec{u}_n)_n$ (the initial values may differ). Hence, the characteristic polynomial of each of them is $\psi(x) = x^s - c_{s-1}x^{s-1} - \cdots - c_1 - c_0$, with roots $\alpha_1, \ldots, \alpha_\ell$. If we suppose that this polynomial has a dominant root, then we can write (for all $1 \leq i \leq d$):

$$u_n^{(i)} = a_i\alpha_1^n + O(n^s\beta^k),$$

where $a_i \in \mathbb{C}$ and $|\alpha| > |\beta| = \max_{2\leq j\leq \ell}\{|\alpha_j|\}$. Therefore, if $u_n^{(i)} = (x_n^{(i)}, y_n^{(i)}) \in \mathbb{R}^2$, we have:

$$\begin{aligned} \vec{u}_n &= \sum_{i=1}^{d} x_n^{(i)} e_{2i-1} + \sum_{i=1}^{d} y_n^{(i)} e_{2i} \\ &= \alpha^n\vec{v} + O(n^s\beta^n)\vec{w}, \end{aligned} \tag{6}$$

where $\vec{v} = (a_1, \ldots, a_d), \vec{w} = (1, \ldots, 1)$ and $\{e_1, \ldots, e_{2d}\}$ is the standard basis of $\mathbb{R}^{2d} \cong \mathbb{C}^d$. The validity of Identity (6) follows from the fact that $x_n^{(i)}$ and $y_n^{(i)}$ have the same asymptotic order (thus, $x_n^{(i)} \asymp y_n^{(i)}$). Since $\|w\| = \sqrt{2d} = O(1)$, we can simply mimic the proof of Theorem 1 to obtain the following result (with notation as above):

Theorem 2. *Let $(\vec{u}_n)_n$ be a linear recurrence in $\mathbb{C}^d$ with a simple dominant root (i.e., $\psi(x)$ has a simple dominant root). Let $P(z_1, \ldots, z_d) \in \mathbb{C}[z_1, \ldots, z_d]$ be a polynomial with a degree of at least one in each variable z_i ($1 \leq i \leq d$). Then, the sequences:*

$$\left(\left(\sum_{k=n}^{\infty} \frac{1}{P(\vec{u}_k)}\right)^{-1}\right)_n \text{ and } (P(\vec{u}_n) - P(\vec{u}_{n-1}))_n$$

are asymptotically equivalent (here, as usual, $P(\vec{u}_k)$ means $P(u_n^{(1)}, \ldots, u_n^{(d)})$).

5. Conclusions

In this paper, we found a sequence which is asymptotically equivalent to partial infinite sums, including a reciprocal of polynomial applied to linear higher order recurrences with a simple dominant root. Further, we included in this article a possible generalization of our main result to a higher dimensional recurrence, which the reviewer pointed out to us.

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