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On Homogeneous Combinations of Linear Recurrence Sequences

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Abstract: Let $(F_n)_{n\geq 0}$ be the Fibonacci sequence given by $F_{n+2} = F_{n+1} + F_n$, for $n \geq 0$, where $F_0 = 0$ and $F_1 = 1$. There are several interesting identities involving this sequence such as $F_n^2 + F_{n+1}^2 = F_{2n+1}$, for all $n \geq 0$. In 2012, Chaves, Marques and Togbé proved that if $(G_m)_m$ is a linear recurrence sequence (under weak assumptions) and $G_{n+1}^s + \cdots + G_{n+\ell}^s \in (G_m)_m$, for infinitely many positive integers n, then s is bounded by an effectively computable constant depending only on ł and the parameters of $(G_m)_m$. In this paper, we shall prove that if $P(x_1, \ldots, x_\ell)$ is an integer homogeneous s-degree polynomial (under weak hypotheses) and if $P(G_{n+1}, \ldots, G_{n+\ell}) \in (G_m)_m$ for infinitely many positive integers n, then s is bounded by an effectively computable constant depending only on ℓ, the parameters of $(G_m)_m$ and the coefficients of P.

Keywords: homogeneous polynomial; linear forms in logarithms; linear recurrence sequence

MSC: 11B39; 11J86X

1. Introduction

A sequence $(G_n)_{n\geq 0}$ is a linear recurrence sequence with coefficients $c_0, c_1, \ldots, c_{k-1}$, with $c_0 \neq 0$, if

$$G_{n+k} = c_{k-1}G_{n+k-1} + \cdots + c_1 G_{n+1} + c_0 G_n, \tag{1}$$

for all positive integer n. A recurrence sequence is therefore completely determined by the *initial values* $G_0, \ldots, G_{k-1}$, and by the coefficients $c_0, c_1, \ldots, c_{k-1}$. The integer k is called the *order* of the linear recurrence. The *characteristic polynomial* of the sequence $(G_n)_{n\geq 0}$ is given by

$$G(x) = x^k - c_{k-1}x^{k-1} - \cdots - c_1 x - c_0 = (x - r_1)^{m_1} \cdots (x - r_t)^{m_t},$$

where $r_1, \ldots, r_t$ are called the root of the recurrence. A root r_j of the recurrence is called a *dominant root* if $|r_j| > |r_i|$, for all $j \neq i \in \{1, 2, \ldots, t\}$. A fundamental result in the theory of recurrence sequences asserts that there exist uniquely determined non-zero polynomials $g_1, \ldots, g_t \in \mathbb{Q}(\{r_j\}_{j=1}^t)[x]$, with $\deg g_j \leq m_j - 1$, for $j = 1, 2, \ldots, t$, such that

$$G_n = g_1(n)r_1^n + \cdots + g_t(n)r_t^n. \tag{2}$$

For a proof of (2), see (Reference [1] Theorem C.1). Also, if r_1 is the dominant root, then $g_1(n)$ is called the dominant polynomial.

The well-known Fibonacci sequence is a recurrence of order 2 given by the recursion

$$F_{n+2} = F_{n+1} + F_n, \text{ with } F_0 = 0 \text{ and } F_1 = 1.$$

The Fibonacci numbers are known for their amazing properties (see Reference ([2] pp. 53–56) and References [3–7]). For example, we have

$$F_n^2 + F_{n+1}^2 = F_{2n+1}, \text{ for all } n \geq 0. \tag{3}$$

In 2010, Marques and Togbé [8] searched for similar identities in higher powers. They proved that if $F_n^s + F_{n+1}^s$ is a Fibonacci number for all sufficiently large n, then $s = 1$ or 2 (see [9] for a generalization). In 2012, Chaves, Marques and Togbé [10] showed that if $(G_m)_m$ is a linear recurrence sequence (under weak assumptions) and $G_{n+1}^s + \cdots + G_{n+\ell}^s \in (G_m)_m$, for infinitely many positive integers n, then s is bounded by an effectively computable constant depending only on ℓ and the parameters of G_m. Note that $P(x_1, x_2, \ldots, x_\ell) = x_1^s + \cdots + x_\ell^s$ is a homogeneous polynomial.

For any $\ell \geq 2$, we want to study the Diophantine equation

$$P(G_{n+1}, \ldots, G_{n+\ell}) = G_m \tag{4}$$

in positive integers n and m, where $P(x_1, \ldots, x_\ell)$ is an homogeneous polynomial with integer coefficients.

In a first view of this equation, we can use some deep techniques in number theory such as results on finiteness of non-degenerate solutions to S-unit equations (see for instance Reference [11] and the references therein) to ensure that m should be a linear function in n, say $an + b$. Now, we can apply the Skolem-Mahler-Leech theorem to ensure that if

$$P(G_{n+1}, \ldots, G_{n+\ell}) - G_{an+b} = 0$$

has infinitely many solutions, then they lie (with the possible exception of finitely many of them) in a union of arithmetic progressions.

The goal of this work is to use more elementary tools to prove that if Equation (4) has infinitely many solutions, then the degree of P must be bounded for an effectively computable constant. This result, in particular, generalizes the main result of Reference [8]. More precisely, we have

Theorem 1. *Let $(G_n)_n$be an integer linear recurrence sequence having dominant root, which is multiplicatively independent with the dominant polynomial. Also, let $P(x_1, \ldots, x_\ell) = Q(x_1, \ldots, x_{\ell-1}) + x_\ell^s$ be an integer homogeneous polynomial of degree s (with $\ell > 1$). If Equation (4) has infinitely many solutions in positive integers m and n, then there exists an effectively computable constant C such that $s < C$. The constant C depends only on ℓ, the parameters of $(G_m)_m$ and the coefficients of P.*

Our main approaches to the proof lie in some asymptotic estimates together with lower bounds for linear forms in logarithms (Baker's method), this method was used for example, in References [12–19].

2. Auxiliary Results

Before proceeding further, we shall need some key results. The first tool is a lower bound for a linear form in logarithms *à la Baker*. From the main result of Matveev [20], we deduce the following lemma.

Lemma 1. *Let $\gamma_1, \ldots, \gamma_t$ be real algebraic numbers and let $b_1, \ldots, b_t$ be nonzero rational integer numbers. Let D be the degree of the number field $\mathbb{Q}(\gamma_1, \ldots, \gamma_t)$ over $\mathbb{Q}$ and let A_j be a real number satisfying*

$$A_j \geq \max\{Dh(\gamma_j), |\log \gamma_j|, 0.16\}, \text{ for } j = 1, \ldots, t.$$

Assume that

$$B \geq \max\{|b_1|, \ldots, |b_t|\}.$$

If $\gamma_1^{b_1} \cdots \gamma_t^{b_t} \neq 1$, *then*

$$|\gamma_1^{b_1} \cdots \gamma_t^{b_t} - 1| \geq \exp(-1.4 \cdot 30^{t+3} \cdot t^{4.5} \cdot D^2(1 + \log D)(1 + \log B)A_1 \cdots A_t).$$

The *logarithmic height* of an s-degree algebraic number γ is defined as

$$h(\gamma) = \frac{1}{s}(\log |a| + \sum_{j=1}^{s} \log \max\{1, |\gamma^{(j)}|\}),$$

where a is the leading coefficient of the minimal polynomial of γ (over $\mathbb{Z}$) and $(\gamma^{(j)})_{1 \leq j \leq s}$ are the conjugates of γ (over $\mathbb{Q}$).

The next lemmas play an important role in the proof of Theorem 1.

Lemma 2. *Let* $(G_n)_n$ *be a linear recurrence whose characteristic polynomial has a simple positive dominant root. Then the dominant polynomial of* $(G_n)_n$ *is a positive constant.*

Proof. We know that (2) holds, where each r_j has multiplicity m_j, and each $g_j(n)$ is a non-zero polynomial with degree $\leq m_j - 1$. Suppose that r_1 is the dominant root, since it is simple, we have immediately $m_1 = 1$ and then the degree of $g_1(n)$ is at most $m_1 - 1 = 0$, so it is a constant, say g_1. Now, dividing G_n by r_1^n, we get

$$0 < \frac{G_n}{r_1^n} = g_1 + \sum_{j=2}^{\ell} \frac{g_j(n)}{\kappa_j^n}, \tag{5}$$

where $\kappa_j = r_1/r_j$. Since $|\kappa_j| > 1$, we have

$$\lim_{n \to \infty} \frac{g_j(n)}{\kappa_j^n} = 0, \text{ for all } 2 \leq j \leq \ell,$$

and so

$$0 \leq \limsup_{n \to \infty} \frac{G_n}{r_1^n} = g_1.$$

Therefore, $g_1 > 0$ as $g_1 \neq 0$. □

Lemma 3. *Let* $(G_n)_n$ *be a linear recurrence with a dominant root. For any nonnegative integers i and r, we have*

$$\lim_{n \to \infty} \frac{G_{n+i}^r}{r_1^{rn}} = g_1^r r_1^{ir}. \tag{6}$$

Proof. For any $i \geq 0$, we use (2) together the multinomial theorem, to obtain

$$G_{n+i}^r = \sum_{\overline{\alpha} \in \mathcal{I}_r} \frac{r!}{\alpha_1! \cdots \alpha_t!} \prod_{j=1}^{t} g_j(n+t)^{\alpha_j} r_1^{\alpha_1(n+i)} \cdots r_t^{\alpha_t(n+i)},$$

where $\mathcal{I}_r = \{\overline{\alpha} = (\alpha_1, \ldots, \alpha_t) \in \mathbb{Z}^t : \alpha_i \geq 0 \text{ and } \sum_{i=1}^{t} \alpha_i = r\}$. A straightforward computation shows that

$$\frac{G_{n+i}^r}{r_1^{rn}} = (g_1(n+i))^s r_1^{ir} + \sum_{\overline{\alpha} \in \mathcal{I}_r \setminus \{r\mathbf{e}_1\}} \frac{r!}{\alpha_1! \cdots \alpha_t!} \prod_{j=1}^{t} g_j(n+t)^{\alpha_j} \frac{r_1^{\alpha_1 i} r_2^{\alpha_2(n+i)} \cdots r_t^{\alpha_t(n+i)}}{r_1^{(r-\alpha_1)n}},$$

where $\mathfrak{e}_1 = (1, 0, \ldots, 0)$. Since $r - \alpha_1 = \alpha_2 + \cdots + \alpha_\ell > 0$ and $g_1(n+i) = g_1$ (by Lemma 2), we can write

$$\frac{G^r_{n+i}}{r_1^{rn}} = g_1^r r_1^{ir} + \sum_{\overline{\alpha} \in \mathcal{I}_r \setminus \{r\mathfrak{e}_1\}} \frac{r!}{\alpha_1! \cdots \alpha_t!} \prod_{j=2}^{t} \left(\frac{g_j(n+i)}{\kappa_j^n} \right)^{\alpha_j} g_1^{\alpha_1} r_1^{\alpha_1 i} r_2^{\alpha_2 i} \cdots r_t^{\alpha_t i},$$

where $\kappa_j = r_1/r_j$. Since $|\kappa_j| > 1$, then $\lim_{n\to\infty} g_j(n+i)/\kappa_j^n = 0$ and so each term in the previous summation tends to zero as $n \to \infty$, because $\alpha_j > 0$, for some $2 \leq j \leq t$. So,

$$\lim_{n\to\infty} \frac{G^r_{n+i}}{r_1^{rn}} = g_1^r r_1^{ir}.$$

□

Now, we are ready to deal with the proof of Theorem 1.

3. The Proof of Theorem 1

Suppose that the pair (n, t_n) is solution of Equation (4) for all n belonging to an infinite set $\mathcal{N}$. Thus, we have $P(G_{n+1}, \ldots, G_{n+\ell}) = G_{t_n}$, for all $n \in \mathcal{N}$. By the formula in (5), we have that $G_m = O(r_1^m)$ (where as usual, O denotes the 'big-oh' Landau symbol). Thus the equality $P(G_{n+1}, \ldots, G_{n+\ell}) = G_{t_n}$ yields $O(r_1^{ns}) = O(r_1^{t_n})$ and so $t_n = O(n)$. Now, we rewrite $P(G_{n+1}, \ldots, G_{n+\ell}) = G_{t_n}$ as

$$\sum_{I \in \mathcal{I}} a_I (G^{i_1}_{n+1} \cdots (G_{n+\ell-1})^{i_{\ell-1}} + (G_{n+\ell})^s = G_{t_n},$$

where the a_I's are the coefficients of P and the sum runs over a subset $\mathcal{I} \subseteq \mathcal{I}_s \subseteq \mathbb{Z}^{\ell-1}$. We divide the above equality by r_1^{sn} and by using that $i_1 + \cdots + i_{\ell-1} = s$, we arrive at

$$\sum_{I \in \mathcal{I}} a_I \prod_{j=1}^{\ell-1} \frac{(G_{n+j})^{i_j}}{r_1^{i_j n}} + \frac{(G_{n+\ell})^s}{r_1^{sn}} = \frac{G_{t_n}}{r_1^{sn}}.$$

Now, when n tends to infinity (in $\mathcal{N}$), we use Lemma 3 to get

$$\lim_{\substack{n\to\infty \\ n\in\mathcal{N}}} \frac{G_{t_n}}{r_1^{ns}} = \sum_{I \in \mathcal{I}} g_1^s r_1^{i_1 + 2i_2 + \cdots + (\ell-1)i_{\ell-1}} + (g_1)^s r_1^{\ell s}. \tag{7}$$

On the other hand,

$$\frac{G_{t_n}}{r_1^{ns}} = g_1 r_1^{t_n - ns} + g_2(t_n) \frac{r_2^{t_n}}{r_1^{ns}} + \cdots + g_t(t_n) \frac{r_t^{t_n}}{r_1^{ns}}.$$

Now, we use that any exponential function of n dominates any polynomial function of n, together with the fact that $t_n = O(n)$ and $|r_1| > |r_j|$, for $j = 2, 3, \ldots, t$, to conclude that

$$\lim_{\substack{n\to\infty \\ n\in\mathcal{N}}} \frac{g_j(t_n)}{r_1^{ns/2}} \cdot \frac{r_j^{t_n}}{r_1^{ns/2}} = 0, \text{ for } j = 2, 3, \ldots, t.$$

So

$$\lim_{\substack{n\to\infty \\ n\in\mathcal{N}}} \frac{G_{t_n}}{r_1^{ns}} = g_1 \lim_{\substack{n\to\infty \\ n\in\mathcal{N}}} r_1^{t_n - ns}.$$

Since $t_n - ns$ is an integer and $r_1 > 1$, then, by (7), $t_n - ns$ must be constant with respect to n, say ν, for all $n \in \mathcal{N}$ sufficiently large. Hence, we have

$$\sum_{I \in \mathcal{I}} a_I g_1^s r_1^{i_1 + 2i_2 + \cdots + (\ell-1)i_{\ell-1}} + (g_1)^s r_1^{\ell s} = g_1 r_1^{\nu}. \tag{8}$$

Now, dividing by $(g_1)^s r_1^{\ell s}$ and after a straightforward calculation, we obtain

$$|(g_1)^{1-s}r_1^{\nu-\ell s}-1| \quad \leq \quad \frac{L(P)}{r_1^{\ell s-(b_1+2b_2+\cdots+(\ell-1)b_{\ell-1})}},$$

where $L(P)$ denotes the length of P (which is the sum of the absolute value of its coefficients) and $(b_1,\ldots,b_{\ell-1})\in\mathcal{I}$ is such that $b_1+2b_2+\cdots+(\ell-1)b_{\ell-1}$ is maximal. Now, we will get a lower bound for $\ell s-(b_1+2b_2+\cdots+(\ell-1)b_{\ell-1})$. In fact, we see that $b_{\ell-1}\leq s, b_{\ell-2}+b_{\ell-1}\leq s,\ldots,b_1+\cdots+b_{\ell-1}\leq s$ and by summing these inequalities, we arrive at $b_1+2b_2+\cdots+(\ell-1)b_{\ell-1}\leq(\ell-1)s$ and so $\ell s-(b_1+2b_2+\cdots+(\ell-1)b_{\ell-1})\geq s$. Therefore,

$$|(g_1)^{1-s}r_1^{\nu-\ell s}-1|\leq\frac{L(P)}{r_1^s}. \tag{9}$$

In order to apply Lemma 1, we can suppose that $t\neq\ell s$ (otherwise, we would have a linear form in one logarithms and the proof would follows in the same way). Then, we choose

$$\gamma_1=g_1,\ \ \gamma_2=r_1,\ \ b_1=1-s,\ \ b_2=\nu-\ell s.$$

Before applying the lemma, let us see that we can suppose that $(g_1)^{1-s}r_1^{\nu-\ell s}\neq 1$. In fact, otherwise we would get a multiplicative relation between g_1 and r_1 (which are multiplicatively independent), this would imply in $s=1$ (and the proof would be done). So, we can suppose that $|(g_1)^{1-s}r_1^{\nu-\ell s}-1|\neq 0$.

Note that $D=[\mathbb{Q}(\gamma_1,\gamma_2):\mathbb{Q}]\leq k!$. Note that the conditions to apply Lemma 1 are fulfilled for A_1,A_2,A_3 depending only of ℓ, the parameters of $(G_m)_m$ and the coefficients of P. Now, we must give an estimate to

$$\max\{|s-1|,|\nu-\ell s|\}\leq|\nu|+\ell s\leq\nu^2+\ell s.$$

If $\nu-\ell s<0$, then $\max\{|s-1|,|\nu-\ell s|\}\leq\ell s(\ell s+1)$. If $\nu-\ell s>0$, we divide (8) by $(g_1)^s r_1^{\ell s}$ and by using that $i_j\leq s$, we obtain

$$|r_1|^{\nu-\ell s}\leq 2|g_1|^{s-1}L(P)|r_1|^{\ell(\ell-1)s/2}.$$

Thus, there is an integer constant $C_1>1$ depending only on $L(P), r_1$ and g_1 such that $\nu-\ell s<C_1 s(s+1)$. Therefore, we can choose $B:=\max\{\ell s(\ell s+1),C_1 s(s+1)\}<4C_1\ell^2 s^4$ (where we used that $x+y\leq xy$, for all positive integers x and y, with $\min\{x,y\}>1$). By Lemma 1, we obtain

$$|(g_1)^{1-s}r_1^{\nu-\ell s}-1|>\exp(-7.7\cdot 10^8(k!)^3(1+\log B)A_1A_2A_3).$$

Since $1+\log B\leq 2\log B<2\log(4C_1\ell^2 s^4)\leq 8\log(4C_1\ell^2)\log s$, then

$$|(g_1)^{1-s}r_1^{\nu-\ell s}-1|>\exp(-62\cdot 10^8(k!)^3 A\log(4C_1\ell^2)\log s), \tag{10}$$

for $A:=A_1A_2A_3$. Now, we combine the estimates (9) and (10), and after a straightforward calculation (e.g., by applying the log function in the inequality), we arrive at

$$\frac{s}{\log s}<\frac{1}{\log r_1}(124\cdot 10^8(k!)^3 A\cdot L(P)\log(4C_1\ell^2)),$$

where we used that $\log L(P)\leq 2L(P)$. Now, let us rewrite the above inequality as

$$\frac{s}{\log s}<M,$$

for M defined as $M := (124 \cdot 10^8 (k!)^3 A \cdot L(P) \log(4C_1 \ell^2))/\log r_1$ (note that this number depends only on ℓ, the parameters of $(G_m)_m$ and the coefficients of P).

On the other side, it is well-known that, for $K > e$,

$$\frac{x}{\log x} < K \quad \text{implies that} \quad x < 2K \log K.$$

In fact, this inequality can be proved by noting that the function $x \to x/\log x$ is increasing for $x > e$. Therefore, we conclude, from $s/\log s < M$, that

$$s < 2M \log M.$$

In conclusion, there exists an effectively computable constant $C := 2M \log M$, which depends only on ℓ, the parameters of $(G_m)_m$ and the coefficients of P, such that $s < C$. □

4. Conclusions

In this paper, we study the values of a homogeneous polynomial at a tuple of consecutive terms of *a linear recurrence sequence* (under some weak technical assumptions), which are still a term of this sequence. More precisely, if $(G_n)_n$ is *a linear recurrence sequence*, we want to study the equation $P(G_{n+1}, \ldots, G_{n+\ell}) = G_m$, where $P(x_1, \ldots, x_\ell)$ is an s-degree homogeneous polynomial with integer coefficients (i.e., $P(\lambda x_1, \ldots, \lambda x_\ell) = \lambda^s P(x_1, \ldots, x_\ell)$, for all $\lambda \in \mathbb{R}$). In particular, we proved that if this equation has infinitely many solutions, then s is bounded for some computable constant depending on ℓ, the parameters of $(G_m)_m$ and the coefficients of P.

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References

1. Shorey, T.N.; Tijdeman, R. *Exponential Diophantine Equations*; Cambridge Tracts in Mathematics 87; Cambridge University Press: Cambridge, UK, 1986.
2. Posamentier, A.S.; Lehmann, I. *The (Fabulous) Fibonacci Numbers*; Prometheus Books: Amherst, NY, USA, 2007.
3. Koshy, T. *Fibonacci and Lucas Numbers with Applications*; Wiley: New York, NY, USA, 2001.
4. Kalman, D.; Mena, R. The Fibonacci Numbers-Exposed. *Math. Mag.* **2003**, *76*, 167–181.
5. Soykan, Y. Tribonacci and Tribonacci-Lucas Sedenions. *Mathematics* **2019**, *7*, 74. [CrossRef]
6. Alfuraidan, M.R.; Joudah, I.N. On a New Formula for Fibonacci's Family m-step Numbers and Some Applications. *Mathematics* **2019**, *7*, 805. [CrossRef]
7. Choo, Y. Relations between Generalized Bi-periodic Fibonacci and Lucas Sequences. *Mathematics* **2020**, *8*, 1527. [CrossRef]
8. Marques, D.; Togbé, A. On the sum of powers of two consecutive Fibonacci numbers. *Proc. Jpn. Acad. Ser. A* **2010**, *86*, 174–176. [CrossRef]
9. Luca, F.; Oyono, R. An exponential Diophantine equation related to powers of two consecutive Fibonacci numbers. *Proc. Jpn. Acad. Ser. A* **2011**, *87*, 45–50. [CrossRef]
10. Chaves, A.P.; Marques, D.A.; Togbé, A. On the sum of powers of terms of a linear recurrence sequence. *Bull. Braz. Math. Soc.* **2012**, *43*, 397–406. [CrossRef]
11. Bugeaud, Y.; Győory, K. Bounds for the solutions of unit equations. *Acta Arith.* **1996**, *74*, 67–80. [CrossRef]
12. Marques, D. The proof of a conjecture concerning the intersection of k-generalized Fibonacci sequences. *Bull. Braz. Math. Soc.* **2013**, *44*, 455-468. [CrossRef]
13. Bravo, J.J.; Luca, F. On a conjecture about repdigits in k-generalized Fibonacci sequences. *Publ. Math. Debr.* **2013**, *82*, 3–4. [CrossRef]

14. Bravo, J.J.; Luca, F. Coincidences in generalized Fibonacci sequences. *J. Number Theory* **2013**, *133*, 2121–2137. [CrossRef]
15. Qu, Y.; Zeng, J.; Cao, Y. Fibonacci and Lucas Numbers of the Form $2^a + 3^b + 5^c + 7^d$. *Symmetry* **2018**, *10*, 509. [CrossRef]
16. Trojovský, P. On Terms of Generalized Fibonacci Sequences which are Powers of their Indexes. *Mathematics* **2019**, *7*, 700. [CrossRef]
17. Trojovský, P. Fibonacci numbers with a prescribed block of digits. *Mathematics* **2020**, *8*, 639. [CrossRef]
18. Ddamulira, M. Repdigits as sums of three Padovan number. *Bol. Soc. Mat. Mex.* **2020**, *26*, 247–261. [CrossRef]
19. Alahmadi, A.; Altassan, A.; Luca, F.; Shoaib, H. Fibonacci numbers which are concatenations of two repdigits. *Quaest. Math.* **2020**. [CrossRef]
20. Matveev, E.M. An explicit lower bound for a homogeneous rational linear form in logarithms of algebraic numbers II. *Izv. Math.* **2000**, *64*, 1217–1269. [CrossRef]