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Axiomatic Definition of Fuzzy Present Value from the Economic Point-view

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Abstract: The concept of present value (PV) may be determined as the cash flow utility which additionally satisfies the conditions: 1° the utility of any current payment is equal to nominal value of this payment; 2° the utility is always an odd function of payment nominal value. Obtained in this way PV definition is called generalized PV definition because of it is more general than the Peccati's axiomatic PV definition. Due behavioral reasons the PV may be imprecisely valued. The fuzzy set theory is applied as effective tool of this imprecision description. Proposed by Calzi's axiomatic definition of fuzzy PV (F-PV) is strictly connected with the Peccati's definition which is weakly consistent with the economics. It is shown that from economical point view the general PV definition is better than the Peccati's one. Thus, we take into account the generalized definition of PV as the starting point for construction of a new axiomatic F-PV definition. All axioms of generalized PV definition are extended to the case when the imprecise present value is given as any fuzzy number. Attached at the end mathematical Appendix contains explicit definitions of all notions of mathematics for fuzzy systems which are applied in this paper.

Keywords: imprecision; present value; fuzzy number

JEL Classification: C65; G39; G40

1. Introduction

The current equivalent value of any cash flow is called the present value (PV) of this cash flow. In financial arithmetic PV is used for discounting money value. The application basis of the financial arithmetic is the interest theory. The financial arithmetic theory is based on axioms formulated by Peccati (1972). Using this theoretical approach Peccati has defined PV as an additive function of payment value. This approach to financial arithmetic was extensively developed in recent years (Janssen et al 2009). In (Piasecki 2012), the Peccati's PV definition is generalized in such way that PV is defined as a utility of the multi-criterial comparison determined by the temporal preference (Mises 1962) and the wealth preference. If PV is defined as utility, then it may be non-additive function of payment value.

The behavioral premises imply that PV may be imprecise. It is widely accepted that imprecise PV is modelled by fuzzy numbers. Such particular PV model is called fuzzy PV (F-PV). In (Piasecki 2014c), the evolution of F-PV notion is described in detail.

An axiomatic F-PV definition is given by Calzi (1990). The Calzi's definition is closely related to the Peccati's PV definition. Hence, the Calzi's definition is valid only for cases when PV is additive function of payment.

The PV additivity is weakly consistent with such basic economic principles as, for example, the Gossen's First Law. For this reason, the main aim of this article is to propose such an axiomatic definition of F-PV which will be valid also for the case when the PV may be non-additive function of payment. The starting point for discussion will be the Piasecki's generalization of Peccati's definitions. An important premise for giving final form to the axiomatic F-PV definition will be experience gained by the author during his research Piasecki (2011a, 2011b, 2013, 2014a, 2014b, 2014c) and Piasecki and Siwek (2015, 2017a, 2018a, 2018b, 2018c, 2018d, 2019).

This paper is organized as follows. Different axiomatic definitions of PV are discussed in Section 2. An axiomatic definition of fuzzy PV is proposed in Section 3. Section 4 contains final conclusions. Applied specific mathematical concepts are explained in the Appendix included in Section 5.

2. Axiomatic Definitions of Present Value

The main task of financial arithmetic is to dynamically assess the value of money. The basic assumption of financial arithmetic is that the value of circulating money increases over time. In general, this assumption is justified by the analysis of the Fisher's equation of exchange (Begg et al 2005). We additionally assume here that the amount of circulating money is constant. This assumption is a typical normative condition. For this reason, the money value taken into account in financial arithmetic is called the normative value of money. A growth process of normative money value is called a capital appreciation process. In economic-financial practice, the increase in the money amount is generally faster than the increase in production volume. Therefore, the real value of money decreases over time. We see that the normative money value is different from the real value of money. This implies a question about the essence of normative value. The answer to this question will explain the essence of the basic functions of financial arithmetic.

Let a fixed set of moments $\mathbf{\Theta} \subseteq [\mathbf{0}, +\infty[$ be given. Each payment is represented by cash flow $(\boldsymbol{t}, \boldsymbol{C}) \in \mathbf{\Phi} = \mathbf{\Theta} \times \mathbb{R}$, where $\boldsymbol{t} \in \mathbf{\Theta}$ is the cash flow moment and $\boldsymbol{C} \in \mathbb{R}$ is the nominal value of this cash flow. Each of these cash flows can be either an executed receivable or matured liability. It is obvious that each nominal value of receivable is non-negative. The debtor's liability is always the creditor's receivable. In this situation, each nominal value of liability is equal to the minus value of corresponding receivable. The set $\mathbf{\Phi}$ is called a payments set. The symbol $\mathbf{\Phi}^+ = \mathbf{\Theta} \times \mathbb{R}_\mathbf{0}^+$ denotes a receivables set.

In financial arithmetic, a future value (FV) is a formal model of normative value. For any payments set $\mathbf{\Phi}$, Peccati (1972) defines FV as a function $\boldsymbol{FV}: \mathbf{\Phi} \longrightarrow \mathbb{R}$ fulfilling the conditions:

$$\forall_{\boldsymbol{C} \in \mathbb{R}}: \quad \boldsymbol{FV}(\mathbf{0}, \boldsymbol{C}) = \boldsymbol{C}, \tag{1}$$

$$\forall_{(\boldsymbol{t},\boldsymbol{C}) \in \mathbf{\Phi}^+} \forall_{\boldsymbol{\Delta t} > \mathbf{0}}: \boldsymbol{FV}(\boldsymbol{t} + \boldsymbol{\Delta t}, \boldsymbol{C}) > \boldsymbol{FV}(\boldsymbol{t}, \boldsymbol{C}), \tag{2}$$

$$\forall_{(\boldsymbol{t},\boldsymbol{C_1}),(\boldsymbol{t},\boldsymbol{C_2}) \in \mathbf{\Phi}}: \quad \boldsymbol{FV}(\boldsymbol{t}, \boldsymbol{C_1}) + \boldsymbol{FV}(\boldsymbol{t}, \boldsymbol{C_2}) = \boldsymbol{FV}(\boldsymbol{t}, \boldsymbol{C_1} + \boldsymbol{C_2}). \tag{3}$$

Peccati uniquely determines PV as the function $\boldsymbol{PV}: \mathbf{\Phi} \longrightarrow \mathbb{R}$, given by the identity

$$\boldsymbol{FV}\big(\boldsymbol{t}, \boldsymbol{PV}(\boldsymbol{t}, \boldsymbol{C})\big) = \boldsymbol{C}. \tag{4}$$

Defined above PV fulfils the conditions

$$\forall_{\boldsymbol{C} \in \mathbb{R}}: \quad \boldsymbol{PV}(\mathbf{0}, \boldsymbol{C}) = \boldsymbol{C}, \tag{5}$$

$$\forall_{(\boldsymbol{t},\boldsymbol{C}) \in \mathbf{\Phi}^+} \forall_{\boldsymbol{\Delta t} > \mathbf{0}}: \boldsymbol{PV}(\boldsymbol{t} + \boldsymbol{\Delta t}, \boldsymbol{C}) < \boldsymbol{PV}(\boldsymbol{t}, \boldsymbol{C}), \tag{6}$$

$$\forall_{(\boldsymbol{t},\boldsymbol{C_1}),(\boldsymbol{t},\boldsymbol{C_2}) \in \mathbf{\Phi}}: \quad \boldsymbol{PV}(\boldsymbol{t}, \boldsymbol{C_1}) + \boldsymbol{PV}(\boldsymbol{t}, \boldsymbol{C_2}) = \boldsymbol{PV}(\boldsymbol{t}, \boldsymbol{C_1} + \boldsymbol{C_2}). \tag{7}$$

The conditions (2) and (6) are equivalent to analogous original conditions of temporal monotonicity used by Peccati. Applied here the descriptions of temporal monotonicity are more convenient for our further considerations.

Peccati proves that if the fixed PV satisfies conditions (5), (6) and (7) then uniquely determined by the identity (4) future value satisfies conditions (1), (2) and (3). This means that the Peccati's axiomatic theory may be equivalently developed on the basis of PV defined as any function $\boldsymbol{PV}: \mathbf{\Phi} \longrightarrow \mathbb{R}$ having the properties (5), (6) and (7). Among other things we have here the following theorem:

Theorem (Peccati 1972): *Any function* $\boldsymbol{PV}: \boldsymbol{\Phi} \longrightarrow \mathbb{R}$ *meets the conditions (5), (6) and (7) if it is given by the identity*

$$\boldsymbol{PV(t,C) = C \cdot v(t)}, \tag{8}$$

where the discount factor $\boldsymbol{v}\colon \boldsymbol{\Theta} \to]\mathbf{0};\mathbf{1}]$ *is decreasing time function fulfilling equation*

$$\boldsymbol{v(0) = 1}. \tag{9}$$

For a fixed payment value, the PV function is reduced to the discounted utility notion considered by many researchers. Multithreaded research results on discounted utility are competently discussed in (Doyle 2013). Among other things, there is shown that the Peccati's theory omits the problem of PV dependence on the interaction between payment value and payment time.

The capital synergy effect is that an increase in the capital value implies an increase in the relative appreciation speed. Using the Peccati's theory we over the synergy capital effect (Piasecki 2012).

The diversification principle stats that financial funds should be allocated among different investments. This principle is justified by the portfolio theory (Markowitz 1952). Applying the Peccati theory we ignore the diversification principle (Piasecki 2012).

All the above facts point to weak coherence the Peccati's theory with the theory and practice of economics. Among other things it implies that the interest theory is weakly coherent with economy.

The Peccati's axiomatic approach has been extensively studied by many researchers. The results of these studies are presented in (Janssen et al 2009).

The process of capital appreciation is described above by the axiom (2). Despite this, the Peccati's theory does not explain the capital appreciation phenomenon. This explanation was obtained by showing that any payment PV is equal to the utility of this payment (Piasecki, 2012). Then the PV is defined as any function $\boldsymbol{PV}\colon \boldsymbol{\Phi} \longrightarrow \mathbb{R}$ satisfying the conditions (5), (6) and

$$\forall_{(t,C)\in\Phi}\forall_{\Delta C>0}\colon \quad \boldsymbol{PV(t,C) < PV(t,C+\Delta C)} \ . \tag{10}$$

$$\forall_{(t,C)\in\Phi}\colon \ \boldsymbol{PV(t,-C) = -PV(t,C)} \ , \tag{11}$$

Fulfillment of condition (11) may be obtained via the assumption that the utility of any liability is non-positive. The notion of negative utility is discussed in (Becker et al. 1960), (Cooper et al. 2001) and (Rabin 1993). Presented above definition is a generalization of the Peccati's definition of PV. Therefore, above definition is called generalized one.

In (Piasecki 2015) we can find examples of such PV which is not additive function of payment nominal value. The variability of these PV is fully justified by economic reasons. The significance of generalization the Peccati's definition to the PV generalized definition is shown in this way.

The synergy capital effect and the diversification principle may be considered in the framework of financial arithmetic based on the generalized definition of PV (Piasecki, 2012).

Any payment PV is identical with the utility of the cash flow representing this payment. For this reason, the Gossen's First Law may be considered as an additional PV feature. This law says that the marginal utility of wealth is diminishing (Begg et al, 2005). Therefore, any function $\boldsymbol{PV}\colon \boldsymbol{\Phi} \longrightarrow \mathbb{R}$ fulfills the inequality

$$\forall_{(t,C_1),(t,C_2)\in\Phi^+}\forall_{\alpha\in]0;1[}\colon \ \boldsymbol{\alpha \cdot PV(t,C_1) + (1-\alpha)PV(t,C_2) < PV(t,\alpha C_1 + (1-\alpha)C_2)} \tag{12}$$

The example of PV satisfying the Gossen's first Law is given in (Piasecki, 2015). On the other side each PV satisfying the conditions (1), (2) and (3) does not fulfil the condition (12). As we see the Peccati's theory rejects the Gossen's First Law, too.

All above facts show the usefulness of this new generalized theory of financial arithmetic. The generalized definition of PV is more coherent with economics than the Peccati's definition. Therefore, we take into account the generalized definition of PV as the starting point for our further considerations.

3. Fuzzy Present Value

If PV is determined as payment utility, then it is subjective in its nature. Subjective evaluations always depend on behavioral factors. The behavioral environment states are imprecisely described. Therefore, it is necessary to disclose imprecision in PV estimation. Effective tool of the imprecision description is the fuzzy set theory. Attached at the end Appendix contains explicit definitions of all these concepts of mathematics for fuzzy systems which are used in this Section.

The first axiomatic approach to F-PV was given by Calzi (1990) who takes into account the condition (8) as start-point for his considerations. He assumed that value of future cash flow, flow moment and interest rate are imprecise. All of these values are described by means of fuzzy assessment which is called a fuzzy interval in original Calzi's work. At the first Calzi proved that in this situation the discounting factor is also fuzzy assessment. Calzi finally defines F-PV by applying the Zadeh's extension principle for the condition (8). Obtained in this way F-PV is fuzzy assessment.

Due the condition (8), the Calzi's approach to F-PV is strictly connected with the Peccati's definition which is weakly coherent with the economics. Moreover, the Peccati's axioms (5), (6) and (7) are passed over in the Calzi's definition. These facts show that the reconsideration of the fuzzy PV definition is necessary.

Any imprecise PV should be an approximation of a value determined by the normative PV function. Therefore we consider the financial space $(\boldsymbol{\Phi}, \boldsymbol{PV})$, where $\boldsymbol{PV}: \boldsymbol{\Phi} \longrightarrow \mathbb{R}$ is any fixed function fulfilling the axioms (5), (6), (10) and (11) of generalized PV definition. This function assigns to the cash flow $(\boldsymbol{t}, \boldsymbol{C}) \in \boldsymbol{\Phi}$ its exact normative it is necessary to disclose imprecision in PV estimation. Then imprecise PV may be expressed, as fuzzy number $\widetilde{\boldsymbol{\mathcal{R}}}(\boldsymbol{PV}(\boldsymbol{t}, \boldsymbol{C})) \in \mathbb{F}$. In this way, all imprecise evaluations of PV constitute a family $\boldsymbol{\Xi} = \{\boldsymbol{\mu}(\cdot\,|\boldsymbol{t}, \boldsymbol{C}): (\boldsymbol{t}, \boldsymbol{C}) \in \boldsymbol{\Phi}\} \subset [\boldsymbol{0}; \boldsymbol{1}]^{\mathbb{R}}$ of fuzzy number memberships function. From the point-view of multi-valued logics, for fixed $\boldsymbol{x} \in \mathbb{R}$ the value $\boldsymbol{\mu}(\boldsymbol{x}|\boldsymbol{t}, \boldsymbol{C})$ can be considered as truth value of the sentence

$$\boldsymbol{x} = \boldsymbol{PV}(\boldsymbol{t}, \boldsymbol{C}) \ . \tag{13}$$

Therefore the value $\boldsymbol{\mu}(\boldsymbol{x}|\boldsymbol{t}, \boldsymbol{C})$ is called the fulfilment degree the above equality.

The conditions (250 and (26) imply that any membership function $\boldsymbol{\mu}(\cdot\,|\boldsymbol{t}, \boldsymbol{C}): \mathbb{R} \longrightarrow [\boldsymbol{0}; \boldsymbol{1}]$ fulfils the conditions

$$\boldsymbol{\mu}(\boldsymbol{PV}(\boldsymbol{t}, \boldsymbol{C})|\boldsymbol{t}, \boldsymbol{C}) = \boldsymbol{1}, \tag{14}$$

$$\forall_{\boldsymbol{x},\boldsymbol{y},\boldsymbol{z} \in \mathbb{R}}: \ \boldsymbol{x} \le \boldsymbol{y} \le \boldsymbol{z} \Rightarrow \boldsymbol{\mu}(\boldsymbol{y}|\boldsymbol{t}, \boldsymbol{C}) \ge \boldsymbol{min}\{\boldsymbol{\mu}(\boldsymbol{x}|\boldsymbol{t}, \boldsymbol{C}), \boldsymbol{\mu}(\boldsymbol{z}|\boldsymbol{t}, \boldsymbol{C})\}. \tag{15}$$

When we are aiming to replace the exact normative assessment $\boldsymbol{PV}(\boldsymbol{t}, \boldsymbol{C})$ by its approximation, then for the family of membership functions $\boldsymbol{\Xi}$ we impose the conditions which are the generalization axioms (5), (6), (10) and (11) to the fuzzy case.

The condition (5) may be written in an equivalent way as follows

$$\forall_{\boldsymbol{C} \in \mathbb{R}}: \ \ \boldsymbol{\mu}(\boldsymbol{x}|\boldsymbol{0}, \boldsymbol{C}) = \begin{cases} \boldsymbol{1} & \boldsymbol{x} = \boldsymbol{C} \\ \boldsymbol{0} & \boldsymbol{x} \neq \boldsymbol{C} \end{cases}. \tag{16}$$

Using the Zadeh's extension principle, we generalize the condition (11) to the condition

$$\forall_{(\boldsymbol{t}, \boldsymbol{C}) \in \boldsymbol{\Phi}}: \ \boldsymbol{\mu}(\boldsymbol{x}|\boldsymbol{t}, -\boldsymbol{C}) = \boldsymbol{\mu}(-\boldsymbol{x}|\boldsymbol{t}, \boldsymbol{C}). \tag{17}$$

Unlike thing happens with the conditions (6) and (10) determining the PV, as the discounted utility of cash flow. These conditions can be replaced respectively by the inequalities

$$\forall_{(\boldsymbol{t}, \boldsymbol{C}) \in \boldsymbol{\Phi}^{+}} \forall_{\boldsymbol{\Delta t} > \boldsymbol{0}}: \ \ \widetilde{\boldsymbol{\mathcal{R}}}(\boldsymbol{PV}(\boldsymbol{t} + \boldsymbol{\Delta t}, \boldsymbol{C})) \prec \widetilde{\boldsymbol{\mathcal{R}}}(\boldsymbol{PV}(\boldsymbol{t}, \boldsymbol{C})), \tag{18}$$

$$\forall_{(\boldsymbol{t}, \boldsymbol{C}) \in \boldsymbol{\Phi}} \forall_{\boldsymbol{\Delta C} > \boldsymbol{0}}: \ \ \widetilde{\boldsymbol{\mathcal{R}}}(\boldsymbol{PV}(\boldsymbol{t}, \boldsymbol{C})) \prec \widetilde{\boldsymbol{\mathcal{R}}}(\boldsymbol{PV}(\boldsymbol{t}, \boldsymbol{C} + \boldsymbol{\Delta C})) \ . \tag{19}$$

Each of the above-described relationship is a fuzzy relation. It means the necessity of taking this into account when the conditions (6) and (10) are extended to the fuzzy case. To meet this postulate we apply the membership function $\boldsymbol{\nu}_{\prec}: [\boldsymbol{\mathcal{F}}(\mathbb{R})]^{\boldsymbol{2}} \rightarrow [\boldsymbol{0}, \boldsymbol{1}]$ of relations used in the inequalities (18) and (19). This function is explicitly determined by (30).

For any fixed pair $(\boldsymbol{t}, \boldsymbol{C}) \in \boldsymbol{\Phi}^{+}$, we consider the function $\boldsymbol{f_t}: \mathbb{R}^{+} \longrightarrow [\boldsymbol{0}; \boldsymbol{1}]$ determined as follows

$$\boldsymbol{f_t}(\boldsymbol{\Delta t}) = \boldsymbol{\nu}_{\prec}\left(\widetilde{\boldsymbol{\mathcal{R}}}(\boldsymbol{PV}(\boldsymbol{t} + \boldsymbol{\Delta t}, \boldsymbol{C})), \widetilde{\boldsymbol{\mathcal{R}}}(\boldsymbol{PV}(\boldsymbol{t}, \boldsymbol{C}))\right) \tag{20}$$

For fixed pair $(\boldsymbol{t}, \boldsymbol{C}) \in \boldsymbol{\Phi}^{+}$ the value $\boldsymbol{f_t}(\boldsymbol{\Delta t})$ can be considered as truth value of the sentence (18). Therefore the value $\boldsymbol{f_t}(\boldsymbol{\Delta t})$ is called the degree of fulfilment this inequality. We can expect that with the increase in change $\boldsymbol{\Delta t}$ the degree of fulfilment the inequality (18) is not decreasing. According to (33), this condition is equivalent to the condition:

[A] *For any fixed pair* $(\boldsymbol{t}, \boldsymbol{C}) \in \boldsymbol{\Phi}^{+}$ *the function* $\boldsymbol{g_t}: \mathbb{R}^{+} \longrightarrow [\mathbf{0}; \mathbf{1}]$ *given by the identity*

$$\boldsymbol{g_t}(\boldsymbol{\Delta t}) = \mathbf{sup}\{\mathbf{min}\{\boldsymbol{\mu}(\boldsymbol{x}|\boldsymbol{t} + \boldsymbol{\Delta t}, \boldsymbol{C}), \boldsymbol{\mu}(\boldsymbol{x}|\boldsymbol{t}, \boldsymbol{C})\}: \boldsymbol{x} \in \mathbb{R}\} \tag{21}$$

is non- increasing function.

In the crisp case, the condition [A] is reduced to the condition (6). All of the above insights cause that the condition [A] may act out as extension of the condition (6) to the fuzzy case.

For any fixed pair $(\boldsymbol{t}, \boldsymbol{C}) \in \boldsymbol{\Phi}$ let us consider the function $\boldsymbol{f_C}: \mathbb{R}^{+} \longrightarrow [\mathbf{0}; \mathbf{1}]$ given by the identity

$$\boldsymbol{f_C}(\boldsymbol{\Delta C}) = \boldsymbol{\nu}_{\prec}\left(\widetilde{\boldsymbol{\mathcal{R}}}(\boldsymbol{PV}(\boldsymbol{t}, \boldsymbol{C})), \widetilde{\boldsymbol{\mathcal{R}}}(\boldsymbol{PV}(\boldsymbol{t}, \boldsymbol{C} + \boldsymbol{\Delta C}))\right) \tag{22}$$

From the point-view of multi-valued logics, for fixed $(\boldsymbol{t}, \boldsymbol{C}) \in \boldsymbol{\Phi}$ the value $\boldsymbol{f_C}(\boldsymbol{\Delta C})$ can be considered as truth value of the sentence (19). Therefore the value $\boldsymbol{f_C}(\boldsymbol{\Delta C})$ is called the degree of fulfilment this inequality. We can expect that with the increase in change $\boldsymbol{\Delta C}$ the degree of fulfilment the inequality (19) is not decreasing. According to (33), this condition is equivalent to the condition:

[B] *For any fixed pair* $(\boldsymbol{t}, \boldsymbol{C}) \in \boldsymbol{\Phi}$ *the function* $\boldsymbol{g_C}: \mathbb{R}^{+} \longrightarrow [\mathbf{0}; \mathbf{1}]$ *given by the identity*

$$\boldsymbol{g_C}(\boldsymbol{\Delta C}) = \mathbf{sup}\{\mathbf{min}\{\boldsymbol{\mu}(\boldsymbol{x}|\boldsymbol{t}, \boldsymbol{C}), \boldsymbol{\mu}(\boldsymbol{x}|\boldsymbol{t}, \boldsymbol{C} + \boldsymbol{\Delta C})\}: \boldsymbol{x} \in \mathbb{R}\} \tag{23}$$

is non-increasing function.

In the crisp case of non-fuzzy PV, the condition [B] is reduced to the condition (10). All of the above insights cause that the condition [B] may act out as extension of the condition (10) to the fuzzy case. In this way we define F-PV as function $\widetilde{\boldsymbol{PV}}: \boldsymbol{\Phi} \longrightarrow \boldsymbol{\mathcal{F}}(\mathbb{R})$ given by the identity

$$\widetilde{\boldsymbol{PV}}(\boldsymbol{t}, \boldsymbol{C}) = \widetilde{\boldsymbol{\mathcal{R}}}(\boldsymbol{PV}(\boldsymbol{t}, \boldsymbol{C})), \tag{24}$$

where $\widetilde{\boldsymbol{\mathcal{R}}}(\boldsymbol{PV}(\boldsymbol{t}, \boldsymbol{C}))$ is such fuzzy number that its membership function $\boldsymbol{\mu}(\cdot\,|\boldsymbol{t}, \boldsymbol{C}): \mathbb{R} \longrightarrow [\mathbf{0}; \mathbf{1}]$ fulfills the conditions (14), (15), (16), (17) [A] and [B].

4. Concluding Remarks

Proposed in this paper axiomatic definition of F-PV is a strong generalization of Peccati's F-PV definition. Thanks to this generalization, we have been able to include the following effects in any particular F-PV definition (Piasecki 2012):

- the phenomenon of capital synergy effect,
- the phenomenon of diminishing marginal utility of wealth,
- the principle of portfolio diversification.

When using the Peccati F-PV definition, these effects had to be ignored.

In general, the imprecision of F-PV is an image of behavioral aspects of financial funds evaluation. In line with the uncertainty principle (Mises 1961, Kaplan et al., 1967), any anticipated future value is always uncertain. Thus FV is usually described as a random variable. In this situation, any return rate is determined as fuzzy probabilistic set (Hirota 1981). Thanks to this, it is possible to combine subjective premises for the assessment of financial funds with extensive empirical knowledge of financial market. In this way we can simultaneously take into account the subjective and empirical premises for investment-making. When a return rate is described by a fuzzy probabilistic set then it is burdened with a composition of imprecision risk and non-knightian uncertainty risk. The concept such defined return rate may be a starting point for the development of a general theory of financial markets at imprecision risk. Elements of such theory are formulated in Piasecki (2011b, 2014) for the case of F-PV defined as any fuzzy number. Application the conditions (16), (17) [A] and [B] of the F-PV definition will make the thesis of this theory more specific. Then we can obtain new original conclusions. All this will allow us to propose and explore new investment strategies.

Model of F-PV may be used not only to determine return rate as fuzzy probabilistic set. This model may be applied wherever their fuzzy evaluation of PV is used. Examples of such applications can be found in the works (Boussabaine and Elhag 1999), (Chiu and Park 1994), (Fang Yong et al. 2008), (Huang, 2007a, b) and (Haifeng et al 2012).

Summing up, proposed in this article axiomatic definition of F-PV is a significant contribution to the development of a financial markets formal theory.

5. Appendix

The symbol $\mathcal{F}(\mathbb{R})$ indicates the family of all fuzzy subsets in the real numbers space $\mathbb{R}$. Each fuzzy set $\widetilde{A} \in \mathcal{F}(\mathbb{X})$ jest described by means of its membership function $\mu_A : \mathbb{R} \to [0;1]$.

Any subset $\widetilde{A} \in \mathcal{F}(\mathbb{R})$ represents the imprecise estimation. In addition, if its membership function $\mu_A \in [0,1]^{\mathbb{R}}$ satisfies the condition

$$\forall_{x,y,z\epsilon\mathbb{R}}: \ x \leq y \leq z \Rightarrow \mu_A(y) \geq \min\{\mu_A(x), \mu_A(z)\}, \tag{25}$$

then this imprecise estimation is called fuzzy assessment. The space of all fuzzy assessments is denoted by the symbol $\mathbb{A}$.

Any real number $k\epsilon\mathbb{R}$ may be approximately estimated using fuzzy number $\widetilde{\mathcal{R}}(k)$ (Dubois and Prade 1979) which is a particular case of fuzzy assessment represented by its membership function $\mu(\cdot\,|k): \mathbb{R} \to [0;1]$ fulfilling the condition

$$\mu(k|k) = 1 \ . \tag{26}$$

The space of all fuzzy numbers is denoted by the symbol $\mathbb{F}$. It is obvious that we have $\mathbb{F} \subset \mathbb{A}$.

Fuzzy assessments can be used to ordering objects represented by these assessments. At the outset, we define the order relations on the set of all fuzzy assessments. These order relations are defined using only the Zadeh's extension principle.

Let us take into account the assessments $\widetilde{A}, \widetilde{B}\epsilon\mathbb{A}$ represented respectively by their membership functions $\mu_A, \mu_B \epsilon [0,1]^{\mathbb{R}}$. Fuzzy preorder $\preccurlyeq$ is defined by the equivalence

$$\widetilde{A} \preccurlyeq \widetilde{B} \Leftrightarrow \text{„The assessment } \widetilde{A} \text{ is less than or equal to the assessment } \widetilde{\mathbf{B}}\text{''}. \tag{27}$$

The preorder $\preccurlyeq$ is determined by its membership function $\nu_{\preccurlyeq}: [\mathcal{F}(\mathbb{R})]^2 \to [0,1]$ given by the identity

$$\nu_{\preccurlyeq}(\widetilde{A}, \widetilde{B}) = \sup\{\min\{\mu_A(x), \mu_B(y)\}: x \leq y\}. \tag{28}$$

If all compared fuzzy assessments are fuzzy numbers, then above preorder is identical with fuzzy preorder determined by Orlovsky (1978).

Fuzzy strict order $\prec$ is defined by the equivalence

$$\widetilde{A} \prec \widetilde{B} \Leftrightarrow \text{„The assessment } \widetilde{A} \text{ is less than the assessment } \widetilde{\mathbf{B}}\text{''}. \tag{29}$$

The strict $\prec$ is determined by its membership function $\nu_{\prec}: [\mathcal{F}(\mathbb{R})]^2 \to [0,1]$ given by the identity

$$\nu_{\prec}(\widetilde{A}, \widetilde{B}) = \min\{\nu_{\preccurlyeq}(\widetilde{A}, \widetilde{B}), 1 - \nu_{\preccurlyeq}(\widetilde{B}, \widetilde{A})\}. \tag{30}$$

Let us consider the pair of real numbers $k, l \in \mathbb{R}$ such that $k < l$. We take into account a pair of fuzzy numbers $\widetilde{\mathcal{R}}(k), \widetilde{\mathcal{R}}(l)$ determined respectively by their membership functions $\mu(\cdot\,|k)$, $\mu(\cdot\,|l) \in [0;1]^{\mathbb{R}}$. From the condition (26) we obtain

$$1 \geq \sup\{\min\{\mu(x|k), \mu(y|l)\}: x \leq y\} \geq \min\{\mu(k|k), \mu(l|l)\} = 1, \tag{31}$$

On the other side, from the condition (25) we have

$$\sup\{\min\{\mu(x|l), \mu(y|k)\}: x \leq y\} = \sup\{\min\{\mu(x|l), \mu(x|k)\}: x \in \mathbb{R}\}. \tag{32}$$

Due these facts we can say that

$$\nu_{\prec}\left(\widetilde{\mathcal{R}}(k), \widetilde{\mathcal{R}}(l)\right) = 1 - \sup\{\min\{\mu(x|l), \mu(x|k)\}: x \in \mathbb{R}\}. \tag{33}$$

The last identity will be applied for determining the axiomatic definition of F-PV.

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