doi: 10.36689/uhk/hed/2020-01-067



# Nonlinear Vehicle Routing Problem

**Jan PELIKÁN * and Josef JABLONSKÝ**

University of Economics, Prague, Czech Republic; pelikan@vse.cz; jablon@vse.cz
* Correspondence: pelikan@vse.cz

**Abstract:** The article is focused on a new modification of vehicle routing problem (VRP), which differs from linear VRP in two points. The first difference is the objective function which in case of linear VRP expresses the total travel costs whereas in nonlinear VRP it is the travel cost per unit volume that is represented by the nonlinear function being equal to linear-fractional function. The second difference is the set of nodes, which in linear VRP must be involved in the vehicle routes. The set of nodes in nonlinear VRP is divided into mandatory and optional ones. The mandatory nodes must be involved in the vehicle routes, the optional nodes can be either involved in the vehicle routes or neglected. Thus, the objective function of the nonlinear VRP is linear-fractional function. The first step is to linearize this function using Charles-Cooper transformation, and then solve the model using linear programming software. The methods are demonstrated on a numerical example.

**Keywords:** vehicle routing problem; integer programming; linear-fractional object function

**JEL Classification:** C44

## 1. Introduction

Traditional formulation of VRP assumes n nodes, where the first node represents the depot and the remaining ones the customers. The merchandise is transported using the routes starting and ending in the depot. The transport itself is realized by the vehicles with certain capacity and the customer-node requests are given by the volume used for the containment of the requested merchandise in a given vehicle. The route length depends on the order of the nodes of a given route and can be calculated using the distance matrix between each pair of nodes. Instead of the node distance one can also calculate the transport cost of a given vehicle from one node to another. The aim is to minimize the total sum of route distances or eventually to minimize the transport cost of the routes using given vehicles.

The solution procedure of VRP must ensure the following two conditions:

a) All nodes are included at least in one of the routes.

b) The sum requests of all nodes of a route must not exceed the capacity of the vehicle for this route.

This problem can be formulated as integer linear programming model and solved using appropriate software tools. VRP and in general, linear integer programming problems, belong among NP hard problems, i.e. if the number of nodes is higher, in reality more than approx. 30 nodes, it is impossible to obtain optimal solution using standard LP integer solvers (branch and bound method) in a reasonable time. Except for mathematical models one can use heuristic methods such as nearest neighborhood method, insert method or savings method, which can help to obtain a suboptimal solution in reasonable time.

There are many modifications of the conventional form of VRP, which arise as a results of merchandise transport in praxis (Laporte 1992). The following ones belong among the most interesting ones - VRP involving vehicles with different capacities and transport cost; VRP with more than one depot; split delivery VRP; VRP involving stochastic demand in nodes; VRP with time windows, where the time of vehicle arrival in the node must be inside to a certain time interval denoted as time window (Braysy and Gendreau 2005), (Desrochers et al. 1992), and others.

The traditional VRP problem is described in the literature enough, heuristic methods are proposed, and a survey of these approaches is summarized in (Laporte 1992). Nonlinear vehicle routing problem (NVRP) we propose in this article differs from the classical one in two main points:

a) The set of nodes is divided into two subsets consisting of either mandatory or optional nodes. The mandatory nodes along the designed vehicle route need to be served. We assume at least one of the nodes to belong to the subset of mandatory nodes.

b) The objective function of the model is non-linear.

NVRP is based on a real case study, in which the career has many orders with some of them being urgent, i.e. taking priority, while the others can be postponed unless the maximal expected delivery time is not exceeded. The order becomes a target of higher priority if its delivery time cannot be extended anymore and thus must be delivered to the customer as soon as possible. Given the fact that every day there are new orders coming up we might come across a situation when the number of items on the list of orders will grow as well and therefore it will be efficient to include the delayed orders in designed routes. On the other hand, if we involve all of the optional nodes in the routes the total route distance and the use of vehicle capacity might become inefficient.

If we assume the objective function is of standard type, i.e. the total sum of all route distances, then the optimal solution would not involve the optional nodes as that would increase the objective function itself. In case we chose the objective function to be represented by the total transported volume, which needs to be maximized, on the contrary the optimal solution would involve all optional nodes, which as noted before would lead to inefficiency in route distance planning as well as in use of vehicle capacity. Based on these reasons we propose a form of objective function that represents the average route distance per transported volume unit. This function, defined as ratio of total route distance and total transported volume, will be minimized. The proposed problem will be first modelled assuming non-linear objective function, that is represented by the linear-fractional function and set of constraints, which is essentially identical to that of classical VRP.

To find the solution for this non-linear problem we use Charles-Cooper transformation that converts our problem to a linear programming one. Despite the fact that the transformation process treats the binary variables the binary condition does not have to hold. Except for the mathematical model one can also modify the heuristic methods designed to solve the classical VRP. Finally, at the end of the article, we present a numerical solution to a problem demonstrating both approaches.

## 2. Mathematical Model of NVRP

*Parameters of the model:*

- $n$ the total number of nodes,
- $m$ the number of optional nodes; assume that nodes $2,3,\ldots,m$ are optional nodes, nodes $m+1, m+2,\ldots, n$ are compulsory nodes, and node 1 is depot,
- $c_{ij}$ the distance between node $i$ and node $j$,
- $q_i$ the demand of node $i$,
- $W$ the capacity of vehicle.

*Variables of the model are:*

- $x_{ij}$ the binary variable with value 1 if a vehicle goes from node $i$ to node $j$, otherwise its value is zero,
- $u_j$ the variables used in anti-cyclic constraints.

The objective function (1) is the ratio where the denominator expresses the total amount of loads of all routes and the numerator is the total length of all routes. Equation (2) ensures that compulsory nodes will be entered and its demand $q_j$ is covered. If the vehicle enters a node it has to leave it – it is ensured by constraints (3). Standard anti-cyclic conditions are in (4). Inequality (5) assures that the capacity of vehicles is not exceeded.

*Mathematical model of NVRP can be written as follows*:

$$f(x) = \frac{\sum_{i=1}^{n} \sum_{j=1}^{n} c_{ij} x_{ij}}{\sum_{i=1}^{n} \sum_{j=1}^{n} q_i\, x_{ij}} \qquad \rightarrow\ min \tag{1}$$

$$\sum_{i=1}^{n} x_{ij} = 1 \quad i = 1,2, \dots, m \tag{2}$$

$$\sum_{i=1}^{n} x_{ij} = \sum_{i=1}^{n} x_{ji} \quad i = 1,2, \dots, n \tag{3}$$

$$u_i + q_j - W\left(1 - x_{ij}\right) \le u_j \quad i = 1,2, \dots, n, j = 2,3, \dots, n, i \neq j \tag{4}$$

$$u_j \le W \quad j = 2,3, \dots, n \tag{5}$$

$$x_{ij} \quad i, j = 1,2, \dots, n, \qquad i \neq j \;\; binary \tag{6}$$

**3. Charnes-Cooper Transformation of Linear Fractional Program to Linear Program**

Model (1)-(6) is not linear in its objective function but can be moved into a linear program rather easily using Charnes-Cooper transformation. Let us assume a general fractional program as follows:

$$g(x) = \frac{c^T x + d}{e^T x + f} \quad \rightarrow \quad min, \tag{7}$$

$$Gx \le h \tag{8}$$

$$Ax = b \tag{9}$$

$$x \ge 0 \tag{10}$$

where $e^T x + f > 0$ for all feasible solutions and the feasible set is nonempty. $x$ is a vector of variables and $G$ and $A$ are metrices. Under these assumptions, the linear fractional program (7)-(10) can be transformed into equivalent linear program (11)- (15) - see e.g. (Martos 1975) and (Barros 1998):

$$g'(x) = c^T x' + d \quad \rightarrow \quad min \tag{11}$$

$$Gx' - h\, t \le 0 \tag{12}$$

$$Ax' - b\, t = 0 \tag{13}$$

$$e^T x' + f\, t = 1 \tag{14}$$

$$x' \ge 0, \qquad t \ge 0 \tag{15}$$

where $x' = \frac{x}{e^T x + f}$ and $t = \frac{1}{e^T x + f}$ .

Now we can apply this transformation to NVRP (1)-(5), i.e. without binary constraint (6). The linear program after this transformation is formulated below - (16)-(21).

$$f'(x) = \sum_{i,j=1}^{n} c_{ij} x'_{ij} \quad \rightarrow \ min \tag{16}$$

$$\sum_{i=1}^{n} x'_{ij} = t \qquad i = 1,2,\dots,m \tag{17}$$

$$\sum_{i=1}^{n} x'_{ij} = \sum_{i=1}^{n} x'_{ji} \quad i = 1,2,\dots,n \tag{18}$$

$$u'_i + \ q_j\, t - W\ t + W x'_{ij} \le u'_j \qquad i = 1,2,\dots,n, j = 2,3,\dots,n, i \ne j \tag{19}$$

$$u'_j \le W\ t \qquad j = 2,3,\dots,n \tag{20}$$

$$x'_{ij} \ \ge 0 \ \ i,j = 1,2,\dots,n, \qquad i \ne j\ , \qquad t \ge 0 \tag{21}$$

where $\quad x'_{ij} = \frac{x_{ij}}{\sum_{i=1}^{n} q_i\, x_{ij}}\ ,\ u'_j = \frac{u_j}{\sum_{i=1}^{n} q_i\, x_{ij}}, t = \frac{1}{\sum_{i=1}^{n} q_i\, x_{ij}}.$

Original variables can be derived as $x_{ij} = \ \ x'_{ij}/\ t$ for all *i, j*. Binary conditions for variables $x_{ij}$ can be ensured by additional conditions (22) and (23). If $x'_{ij}$ = *t*, then $x_{ij}$ = 1, and if $x'_{ij}$ = 0, then $x_{ij}$ = 0.

$$-M(1 - y_{ij}) \le x'_{ij} - t \le \ M(1 - y_{ij}) \quad \ i \ne j \tag{22}$$

$$-M y_{ij} \le x'_{ij} \le \ y_{ij}, \quad i \ne j \tag{23}$$

$$y_{ij}\ \ binary\ \ for\ all \quad i \ne j \tag{24}$$

The mathematical model (16)-(24) is binary linear program and can be solved using conventional LP packages like GUROBI, CPLEX, etc.

**4. Numerical Example**

The proposed mathematical model was verified on an illustrative example. Consider 11 nodes where node 1 is a depot, and the capacity of each vehicle is *W*=100. The requirements of the nodes are *q* = (0 19 24 30 20 35 25 32 20 22 37). The distance matrix *C* is as below:

| | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| 0 | 13 | 6 | 55 | 93 | 164 | 166 | 168 | 169 | 241 | 212 |
| 13 | 0 | 11 | 66 | 261 | 175 | 177 | 179 | 180 | 239 | 208 |
| 6 | 11 | 0 | 60 | 97 | 168 | 171 | 173 | 174 | 239 | 209 |
| 55 | 66 | 60 | 0 | 82 | 113 | 115 | 117 | 117 | 295 | 265 |
| 93 | 261 | 97 | 82 | 0 | 113 | 115 | 117 | 118 | 333 | 302 |
| 164 | 175 | 168 | 113 | 113 | 0 | 6 | 4 | 2 | 403 | 374 |
| 166 | 177 | 171 | 115 | 115 | 6 | 0 | 8 | 7 | 406 | 376 |
| 168 | 179 | 173 | 117 | 117 | 4 | 8 | 0 | 2 | 408 | 378 |
| 169 | 180 | 174 | 117 | 118 | 2 | 7 | 2 | 0 | 409 | 379 |
| 241 | 239 | 239 | 295 | 333 | 403 | 406 | 408 | 409 | 0 | 46 |
| 212 | 208 | 209 | 265 | 302 | 374 | 376 | 378 | 379 | 46 | 0 |

The optimal objective function of model (16)-(24) $f'(x) = 3$ and the otimal value of variable $t = 0.0058$. Therefore, $x'_{ij} = 0.0058$, if $y_{ij} = 1$ otherwise $x'_{ij} = 0$. From optimal values of variables $x'_{ij}$, $y'_{ij}$ $and$ $x_{ij}$ it is possible to derive that the optimal routes are:

1. route: 1-3-2-4-1 with transport volume 73 and length of the route 138,
2. route: 1-5-7-9-6-1 with transport volume 100 and length of the route 381,

The total length of all routes is 519, and the total load is 173. The length on one unit of load is 3 which is the optimal value of the objective function.

## 5. Conclusions

VRP is one of the most discussed optimization problems with variety of real-world applications. Traditional formulation of VRP is linear, i.e. linear objective function and linear set of constraints. In this paper, a new modification of VRP was introduced. This formulation was motivated by real-world study and, in our best of knowledge, it is original and unpublished elsewhere yet. The problem itself is non-linear in its objective function but an original way how to transform it into a linear program was proposed. The solution of the model was illustrated on a simple numerical example. Mathematical model (16)-(24) is hardly solvable for real instances even by using high-quality solvers as GUROBI or CPLEX. Therefore, future research will be focused on solving real examples of this nature using various heuristic methods and on their comparison.

## References

Barros Ana Isabel. 1998. *Discrete and fractional programming techniques for location models*. Dordrecht: Kluwer Academic Publishers. https://doi.org/10.1007/978-1-4615-4072-4_1.

Braysy Olli and Gendreau Michel. 2005. The vehicle routing problem with time windows, Part I. Route construction and local search algorithms. *Transportation Science:* 39(1), 104-118. https://doi.org/10.1287/trsc.1030.0056.

Desrochers Martin, Desrosiers Jacques and Solomon M. Marius, 1992. A new optimization algorithm for the vehicle routing problem with time windows. *Operations Research:* 40, 342-354. https://doi.org/10.1287/opre.40.2.342.

Laporte Gilbert, 1992. The vehicle routing problem: An overview of exact and approximate algorithms. *European Journal for Operations Research:* 59, 345-358. https://doi.org/10.1016/0377-2217(92)90192-C.

Martos Bela. 1975. *Nonlinear programming: Theory and methods*. Amsterdam-Oxford: North-Holland Publishing. https://doi.org/10.1002/zamm.19760561116.