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On Fixed Points of Iterations Between the Order of Appearance and the Euler Totient Function

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Abstract: Let F_n be the nth Fibonacci number. The order of appearance $z(n)$ of a natural number n is defined as the smallest positive integer k such that $F_k \equiv 0 \pmod{n}$. In this paper, we shall find all positive solutions of the Diophantine equation $z(\varphi(n)) = n$, where φ is the Euler totient function.

Keywords: Fibonacci numbers; order of appearance; Euler totient function; fixed points; Diophantine equations

MSC: 11B39; 11DXX

1. Introduction

Let $(F_n)_{n\geq 0}$ be the sequence of *Fibonacci numbers* which is defined by 2nd order recurrence

$$F_{n+2} = F_{n+1} + F_n,$$

with initial conditions $F_i = i$, for $i \in \{0, 1\}$. These numbers (together with the sequence of prime numbers) form a very important sequence in mathematics (mainly because its unexpectedly and often appearance in many branches of mathematics as well as in another disciplines). We refer the reader to [1–3] and their very extensive bibliography.

We recall that an arithmetic function is any function $f : \mathbb{Z}_{>0} \to \mathbb{C}$ (i.e., a complex-valued function which is defined for all positive integer). Some well-known examples are the divisor sum function $\sigma(n)$, prime counting function $\pi(n)$ and the Euler function $\varphi(n)$. The *Euler (totient) function* (which is one of the main topics of this work) is defined as the number of positive integers less than or equal to n and coprime to n, i.e.,

$$\varphi(n) = \#\{k \in [1, n] : \gcd(k, n) = 1\}.$$

The first few values of $\varphi(n)$ are:

$$1, 1, 2, 2, 4, 2, 6, 4, 6, 4, 10, 4, 12, 6, 8, 8, 16, 6, 18, 8, 12, 10, 22, 8, 20.$$

See OEIS [4] sequence A000010, for more values of the φ-function.

Some arithmetic functions are related to some sequences (e.g., the function $\pi(n)$ which is related to prime numbers). For example, an arithmetic function which is related to divisibility properties of Fibonacci numbers has been the main subject of many recent research. In fact, the function $z : \mathbb{Z}_{>0} \to \mathbb{Z}_{>0}$ which is called the *order (or rank) of appearance* of n in the Fibonacci sequence is defined as

$$z(n) = \min\{k \geq 1 : n \mid F_k\},$$

i.e., the smallest positive integer k, such that n divides F_k (see OIES sequence A001177). This function gained a great interest which is reflected in the number of recent works about Diophantine problems which involve it (see [5,6] and its bibliography).

Also, many authors have worked on problems involving arithmetic functions and the Fibonacci sequence. For example, some arithmetic problems involving $\varphi(F_n)$ (as for example, the equation $\varphi(F_n) = 2^a \cdot 3^b$ in [7]) can be found in [8–14] and references therein.

In this paper, we shall search for the fixed points of the composite function $z \circ \varphi$ (see Figure 1). More precisely, our result is the following:

Theorem 1. *The only positive integer solutions of the Diophantine equation*

$$z(\varphi(n)) = n \tag{1}$$

are $n = 1$ *and* $n = 3$.

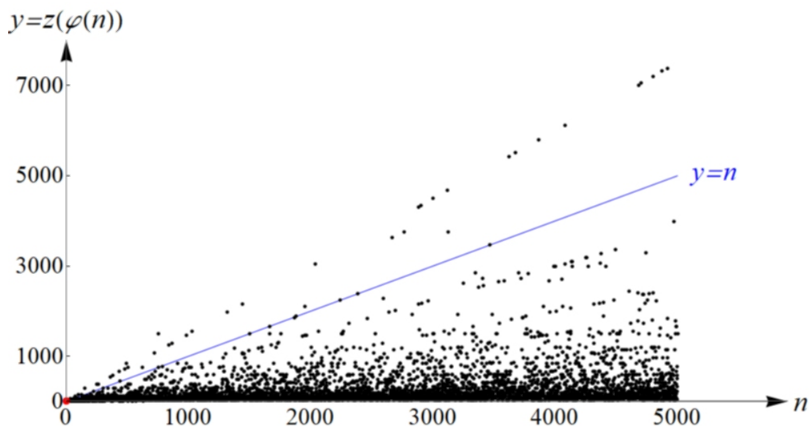

Figure 1. The scatterplot of the function $z(\varphi(n))$ for $n \in [1, 5000]$ (solutions of (1) are colored in red).

We organize the paper as follows. In Section 2, we recall some helpful properties of the functions $\varphi(n)$ and $z(n)$. The last section is devoted to the proof of the theorem.

2. Auxiliary results

Before the proof of Theorem 1, we shall provide two useful lemmas related to the studied functions.

The first result concerns the Euler totient function $\varphi(n)$:

Lemma 1. *We have*

(i) *(multiplicative)* $\varphi(mn) = \varphi(m)\varphi(n)$, *for all coprime integers* m *and* n.
(ii) *(upper bound)* $\varphi(n) \le n$, *for all* $n \ge 1$, *where the equality is attained only for* $n = 1$.
(iii) *(explicit form) If* $n = p_1^{a_1} \cdots p_k^{a_k}$ *is a prime factorization (with* $a_i \ge 1$*), then*

$$\varphi(n) = p_1^{a_1-1}(p_1 - 1) \cdots p_k^{a_k-1}(p_k - 1).$$

(iv) *(parity)* $\varphi(n)$ *is an even number, for all* $n > 2$.

All the previous properties are very well-known and can be easily deduced from the definition of $\varphi(n)$ (however, they can be found in any reasonable number theory book).

The second lemma is about the order of appearance function $z(n)$:

Lemma 2. *We have*

(i) *(minimal value)* $z(n) = 1$ *if and only if* $n = 1$.
(ii) *(general upper bound)* $z(n) \leq 2n$, *for all* $n \geq 1$.
(iii) *(upper bound for primes)* $z(p) \leq p + 1$, *for all prime number* p.
(iv) *(power of* 2*)* $z(2^k) = 3 \cdot 2^{k-2}$, *for all integer* $k \geq 3$.
(v) *(power of* 3*)* $z(3^k) = 4 \cdot 3^{k-1}$, *for all integer* $k \geq 1$.
(vi) *(prime power)* $z(p^k) = p^{k-e_p} z(p)$, *for a prime number* p, *where* $e_p \in [1, k]$.
(vii) *(multiplicative)* $z(mn) = \text{lcm}(z(m), z(n))$, *for all coprime integers* m *and* n.

The previous properties appear in many works on the rank of appearance. So, to be more precise we refer the reader to the recent paper [15] and references therein.

Now, we are ready to deal with the proof of theorems.

3. The Proof of Theorem 1

Clearly $n = 1$ is a solution of Equation (1). Now, assume that $n > 1$ is another integer solution of that equation. Let us consider its prime factorization as

$$n = p_1^{a_1} \cdots p_k^{a_k}, \text{ with } p_1 < \cdots < p_k \text{ and } a_i \geq 1, \text{ for all } i \in [1, k].$$

Our first approach consists in bounding the smallest prime factor of n, i.e., we shall find the possibilities for p_1.

Claim: $p_1 \in \{2, 3\}$. In fact, if $p_1 \neq 2$, then there exists a prime q such that $p_1 \equiv 1 \pmod{q}$. Now, we have that $\varphi(n) = p_1^{a_1-1} \cdots p_k^{a_k-1}(p_1 - 1) \cdots (p_k - 1)$ (Lemma 1 (iii)) and we can write $\varphi(n) = q^r s$, where $r \geq 1$ and $\gcd(q, s) = 1$. So,

$$\begin{aligned} p_1^{a_1} \cdots p_k^{a_k} &= n = z(\varphi(n)) = z(q^r s) \\ &= \text{lcm}(z(q^r), z(s)) = \text{lcm}(q^{r-e} z(q), z(s)), \end{aligned}$$

for some integer $e > 0$ (where we used Lemma 2 (vi) and (vii)). In particular, we have that $z(q)$ divides $p_1^{a_1} \cdots p_k^{a_k}$. Since $z(q) > 1$ (Lemma 2 (i)), we can consider another prime number t such that $t \mid z(q)$. Thus, $t = p_i$, for some $i \in [1, k]$, which yields

$$p_1 \leq p_i = t \leq z(q) \leq q + 1 \leq p_1,$$

where in the last inequality, we used that $q \mid p_1 - 1$ together with Lemma 2 (iii). So, we conclude that $p_1 = q + 1$, where p_1 and q are prime numbers and therefore $p_1 = 3$ (since the only consecutive primes are 2 and 3). This proves the first claim.

Now, the proof conveniently splits into two cases:

The case in which $p_1 = 2$.

In this case, we have that $n = 2^a m$, where m is odd and $a \geq 1$. Thus, $\varphi(2^a m) = 2^{a-1}\varphi(m)$ (by Lemma 1 (i)) and so, we have

$$n = z(\varphi(n)) = z(2^{a-1}\varphi(m)) \leq 2 \cdot 2^{a-1}\varphi(m) \leq 2^a m = n,$$

where we combined Lemma 1 (ii) and Lemma 2 (ii). This yields that $\varphi(m) = m$ and so $m = 1$ (again by Lemma 1 (ii)).

In conclusion, $n = 2^a$ and so,

$$2^a = z(\varphi(2^a)) = z(2^{a-1}) = 1, 3, 6 \text{ or } 2^{a-3},$$

according to $a = 1, 2, 3$ or $a \geq 4$, respectively (here, we applied Lemma 2 (iv)). Therefore, there is no solution in this case.

The case in which $p_1 = 3$.

In this case, we have that $n = 3^a m$, where $\gcd(m, 6) = 1$. Thus, $\varphi(n) = 2 \cdot 3^{a-1}\varphi(m)$. We claim that $m = 1$. In fact, otherwise $m > 4$ (since $\gcd(m, 6) = 1$) and so $\varphi(m)$ is even (by Lemma 1 (iv)), say $\varphi(m) = 2^r s$, with $r \geq 1$ and s odd. Therefore,

$$3^a m = z(\varphi(3^a m)) = z(2^{r+1} \cdot 3^{a-1} s) = \text{lcm}(z(2^{r+1}), z(3^{a-1} s)).$$

However, $r + 1 \geq 1$ and so 2 divides $z(2^{r+1})$ which implies in the absurdity that 2 divides $3^a m$. With this contradiction, we deduce that $m = 1$ and so $n = 3^a$.

Now, we observe that $\varphi(3^a) = 2 \cdot 3^{a-1}$ and hence

$$3^a = z(\varphi(3^a)) = z(2 \cdot 3^{a-1}) = \text{lcm}(z(2), z(3^{a-1})).$$

If $a - 1 \geq 1$, then $z(3^{a-1}) = 4 \cdot 3^{a-2}$ (Lemma 2 (v)) and we would obtain the contradiction of $\text{lcm}(z(2), z(3^{a-1}))$ being an even number (note that the left-hand side above is an odd number). Thus, $a = 1$, that is, $n = 3$. This completes the proof.

4. Conclusions

In this paper, we search for fixed points of the function $z \circ \varphi : \mathbb{Z}_{>0} \to \mathbb{Z}_{>0}$, i.e., we consider the Diophantine equation $z(\varphi(n)) = n$. Here, the functions $z(n) = \min\{k > 0 : n \mid F_k\}$ and $\varphi(n) = \#\{k \in [1, n] : \gcd(n, k) = 1\}$ are the order of appearance (in the Fibonacci sequence) and the Euler (totient) function, respectively. We use some lemmas about properties of these functions to determine the smallest prime factor of n (which is solution of $z(\varphi(n)) = n$) and after that, we complete the proof with some upper bounds for z and φ. We conclude by proving that the only solutions are $n = 1$ and $n = 3$.

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