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Kalman Filter and Time Series

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Abstract. The Kalman filter is one of the classical algorithms of the statistical estimation theory. The filter is applied in a lot of fields. One of them is econometrics, especially its sphere of econometric models in which there is at least one variable which cannot be directly observed and measured. The paper presents the basic features of the Kalman filter and its application in time series analysis. The text specifically focuses on possibilities of transformations of ARMA models into state-space form, and the following application of the Kalman filter in solving problems of prediction, filtering and smoothing. Another issue which is focused on is an application of the Kalman filter in estimating of unknown parameters of time series models. The presented procedures are demonstrated on practical problems which are implemented in the MATLAB environment; the outputs are presented in the text.

Keywords: Kalman Filter, Time Series, ARMA, State-Space Model.

1 Introduction

Numerous models of dynamic systems are designed in the theory of economics [22, 5, 13]. And in the process of applying these models, there are frequent cases of situations when one of the model variables (the system state) is not directly observable, is latent. In such a case it is possible to use the Kalman filter.

The Kalman filter is considered to be a theoretical basis for various recursive methods applied in stochastic (linear) dynamic systems. The algorithm is based on the idea that an unknown state of the system can be estimated using certain measured data (usually in the form of a time series). The algorithm was named after Rudolf Emil Kalman, a Hungarian mathematician living in the USA, who presented it in 1960 in the text referred to below under [11]. During the course of time, other authors derived other algorithms based on the principle of the Kalman filter, these algorithms are generally referred to as Kalman filters, and they can be conveniently applied in specific situations of solving practical problems in which, for example, some of the theoretical assumptions of the classical Kalman filter are not met.

The Kalman filter can be applied in varied domains of the prevailingly technical character, as for example in case of localization of moving objects and navigation – the Kalman filter or Kalman filters in general are used in global navigation satellite systems (GPS, etc.), in radars, in case of navigation and controlling of robots, in autopilots or autonomous vehicles, in computer vision for tracking objects in videos, in augmented and virtual reality, etc. Their application in the sphere of econometrics cannot be ignored [9, 16, 21, 3]; analysis of economic time series can be mentioned here as a related example [14, 20]. The main goal of the paper is to present the basic features of the Kalman filter and focus on its application in time series analysis.

2 Kalman Filter

The Kalman filter is a tool which enables to estimate the state of a stochastic linear dynamic system using measurements corrupted by noise. The estimate produced by the Kalman filter is statistically optimal in some sense (for example when considering the minimization of the mean square error; see [12] for details). The principle of application of the filter is illustrated in Fig. Fig**. 1**.

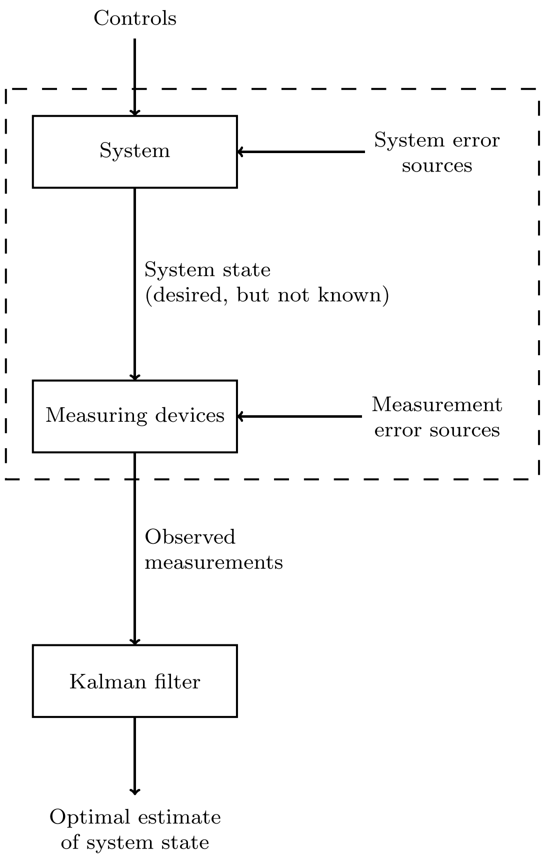

Fig. 1. Scheme of applying Kalman filter. Based on [12].

The Kalman filter works with all available information, i.e. all the available measurements, the knowledge of the system model and the statistical description of its inaccuracies, noise and errors, and the information about the initial conditions are used when the system state is being estimated.

2.1 Algorithm of Kalman Filter

Let us consider a stochastic linear dynamic system in discrete time, which is represented by the following state-space model (it is assumed here that the system has no inputs)

$$\boldsymbol{x}_k = \boldsymbol{\Phi}_{k-1}\boldsymbol{x}_{k-1} + \boldsymbol{G}_{k-1}\boldsymbol{w}_{k-1}, \tag{1}$$

$$\boldsymbol{z}_k = \boldsymbol{H}_k\boldsymbol{x}_k + \boldsymbol{v}_k. \tag{2}$$

The equation (1) referred to as the state equation, describes the dynamics of the system, the vector $\boldsymbol{x}_k \in \mathbb{R}^n$ is an (unknown) vector of the system state at the time t_k, the matrix $\boldsymbol{\Phi}_{k-1} \in \mathbb{R}^{n\times n}$ represents the system state transition between the time t_{k-1} and t_k. The equation (2) is called the measurement equation, the vector $\boldsymbol{z}_k \in \mathbb{R}^m$ is called the system output vector, the measurement vector or the observation vector, the matrix $\boldsymbol{H}_k \in \mathbb{R}^{m\times n}$ describes the relation between the system state and the measurements. Since a stochastic system is concerned, the vectors $\boldsymbol{x}_k$ and $\boldsymbol{z}_k$, $k = 0, 1, 2, \ldots$, can be considered as random variables, and their sequences $\{\boldsymbol{x}_k\}$ and $\{\boldsymbol{z}_k\}$ are then random (stochastic) processes.

$\{\boldsymbol{w}_k\}$ and $\{\boldsymbol{v}_k\}$ are random noise processes; these processes are assumed to be uncorrelated Gaussian processes with zero mean and covariance matrices $\boldsymbol{Q}_k \in \mathbb{R}^{l\times l}$ resp. $\boldsymbol{R}_k \in \mathbb{R}^{m\times m}$ at time t_k (the processes have qualities of Gaussian white noise). Matrix $\boldsymbol{G}_k \in \mathbb{R}^{n\times l}$ then describes the impact of the noise in the state equation of the model.

Furthermore, let us assume that $\boldsymbol{x}_0$ is a random variable having a Gaussian (normal) distribution with known mean $\boldsymbol{x}_0$ and known covariance matrix $\boldsymbol{P}_0$. Moreover, suppose that $\boldsymbol{x}_0$ and both the noises are always mutually uncorrelated. Then we can summarize that for all t_k

$$E\langle \boldsymbol{w}_k \rangle = \boldsymbol{0},$$

$$E\langle \boldsymbol{v}_k \rangle = \boldsymbol{0},$$

$$E\langle \boldsymbol{w}_{k_1}\boldsymbol{w}_{k_2}^T \rangle = \boldsymbol{Q}_{k_1}\Delta(k_2 - k_1),$$

$$E\langle \boldsymbol{v}_{k_1}\boldsymbol{v}_{k_2}^T \rangle = \boldsymbol{R}_{k_1}\Delta(k_2 - k_1),$$

$$E\langle \boldsymbol{w}_{k_1}\boldsymbol{v}_{k_2}^T \rangle = \boldsymbol{0},$$

$$E\langle \boldsymbol{x}_0\boldsymbol{w}_k^T \rangle = \boldsymbol{0},$$

$$E\langle \boldsymbol{x}_0\boldsymbol{v}_k^T \rangle = \boldsymbol{0},$$

where the symbol Δ refers to the Kronecker delta

$$\Delta(k) = \begin{cases} 1, & k = 0, \\ 0, & k \neq 0. \end{cases}$$

The aim of the Kalman filter is to produce an estimate of the state vector $\boldsymbol{x}_k$ at time t_k, symbolized as $\hat{\boldsymbol{x}}_k$, so that this estimate is optimal (for example with respect to minimizing the mean square error).

The algorithm of the Kalman filter is recursive, the calculation at time t_k consists of two main steps. Firstly, the a priori estimate $\hat{\boldsymbol{x}}_{k(-)}$ at time t_k is computed through substituting the a posteriori estimate from time t_{k-1} into the deterministic part of the state equation of the model; this step is called the prediction step. Then, this estimate is improved by using the measurement carried out at time t_k, which results in obtaining the a posteriori estimate $\hat{\boldsymbol{x}}_{k(+)}$ at time t_k; this is the correction step.

The following relation can be written to specify the a priori estimate of the state vector $\hat{\boldsymbol{x}}_{k(-)}$ at time t_k; the uncertainty of this estimate is expressed by the a priori error covariance matrix $\boldsymbol{P}_{k(-)}$

$$\hat{\boldsymbol{x}}_{k(-)} = \boldsymbol{\Phi}_{k-1}\hat{\boldsymbol{x}}_{k-1(+)},$$

$$\boldsymbol{P}_{k(-)} = \boldsymbol{\Phi}_{k-1}\boldsymbol{P}_{k-1(+)}\boldsymbol{\Phi}_{k-1}^T + \boldsymbol{G}_{k-1}\boldsymbol{Q}_{k-1}\boldsymbol{G}_{k-1}^T.$$

Then, after obtaining of the measurement $\boldsymbol{z}_k$, combining of the a priori estimate and the difference between the actual value and the predicted value of the measurement weighted by the matrix $\boldsymbol{K}_k$, we come to the a posteriori estimate of the state vector $\hat{\boldsymbol{x}}_{k(+)}$; its uncertainty is expressed by the a posteriori error covariance matrix $\boldsymbol{P}_{k(+)}$

$$\hat{\boldsymbol{x}}_{k(+)} = \hat{\boldsymbol{x}}_{k(-)} + \boldsymbol{K}_k\left[\boldsymbol{z}_k - \boldsymbol{H}_k\hat{\boldsymbol{x}}_{k(-)}\right],$$

$$\boldsymbol{P}_{k(+)} = \boldsymbol{P}_{k(-)} - \boldsymbol{K}_k\boldsymbol{H}_k\boldsymbol{P}_{k(-)},$$

$$\boldsymbol{K}_k = \boldsymbol{P}_{k(-)}\boldsymbol{H}_k^T\left[\boldsymbol{H}_k\boldsymbol{P}_{k(-)}\boldsymbol{H}_k^T + \boldsymbol{R}_k\right]^{-1}.$$

A detailed derivation of the given equations of the Kalman filter can be find for example in [6], more detailed presentations of the algorithm, its features and its theoretical assumptions can be found for example in [6, 12, 17]; practical aspects of the implementation of the filter are discussed for example in [17].

3 Application of Kalman Filter in Time Series Analysis

The Kalman filter can be conveniently applied when solving the problems of prediction, filtering and smoothing [2, 4, 7, 8]. Prediction is based on the estimation of the system state at certain time while using observations measured at times preceding the time of the estimation; it can be shortly written as $(t_{observation} < t_{estimation})$. Filtering is based on the estimation of the system state at certain time while using observations measured at that given estimation time and preceding times

($t_{observation} \leq t_{estimation}$). Smoothing is based on the estimation of the system state at certain time while using observations measured at times after the time of the estimation ($t_{observation} > t_{estimation}$).

So called ARMA (autoregressive moving average) models or their other more general variants are often used when time series are analyzed. These models, however, can be transformed into the form of the state-space model consisting of the state equation and the measurement equation; the Kalman filter can be then applied to this model. Related to the construction of time series models, the Kalman filter can be further involved in calculating the estimates of the unknown parameters of these models [7, 8].

3.1 Estimation of Parameters of Time Series Models

For the time being, let us consider merely the autoregressive model AR(p) of the p order

$$y_t = \phi_1 y_{t-1} + \phi_2 y_{t-2} + \cdots + \phi_p y_{t-p} + \varepsilon_t,$$

where $\{\varepsilon_t\}$ is an uncorrelated Gaussian process with zero mean and constant variance σ^2, and $\phi_i, i = 1, \ldots, p$, are the parameters of the model. (The used symbols are in accordance with the established practice of time series models.)

Now, the aim can be estimating of the model´s parameters. When solving this problem, the parameters form the unknown state vector, and the whole state equation (supposing that the parameters are constant in time) and the measurement equation can be expressed in the following way

$$\begin{pmatrix} \phi_{1\ t} \\ \phi_{2\ t} \\ \vdots \\ \phi_{p\ t} \end{pmatrix} = \begin{pmatrix} \phi_{1\ t-1} \\ \phi_{2\ t-1} \\ \vdots \\ \phi_{p\ t-1} \end{pmatrix},$$

$$y_t = \begin{pmatrix} y_{t-1} & y_{t-2} & \cdots & y_{t-p} \end{pmatrix} \begin{pmatrix} \phi_{1\ t} \\ \phi_{2\ t} \\ \vdots \\ \phi_{p\ t} \end{pmatrix} + \varepsilon_t.$$

The problem can be illustrated on a simulated AR(2) process

$$y_t = 0.7 y_{t-1} + 0.3 y_{t-2} + \varepsilon_t, \quad \varepsilon_t \sim N(0, 10), \quad t = 1, \ldots, 100,$$

whose possible realization is depicted in Fig Fig. **2**.

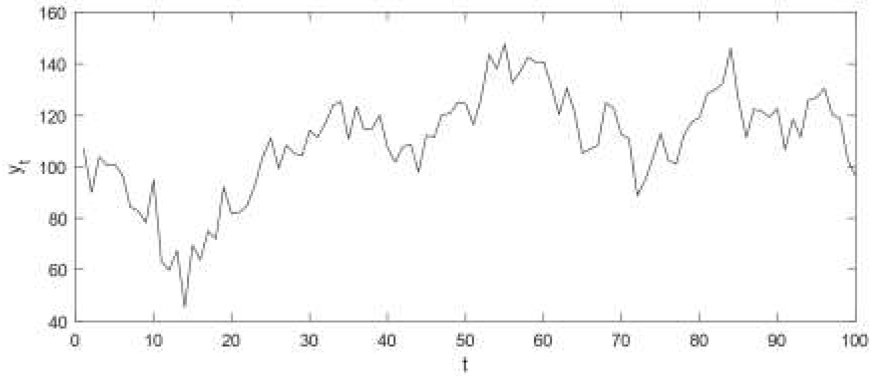

Fig. 2. Simulated AR(2) process.

Only the values y_t are available as the input of the problem, the parameters ϕ_1 and ϕ_2 are unknown. On the basis of the above given, the state equation and the measurement equation were constructed, and the Kalman filter was then applied. The evolution of the obtained estimates of the parameters is summarized in Table Table **1.** Estimates of the parameters of the AR(2) model obtained through Kalman filter. (the initial values of the estimates could be chosen arbitrarily).

Table 1. Estimates of the parameters of the AR(2) model obtained through Kalman filter.

t	0	10	20	30	40	50	60	70	80	90	100
$\hat{\phi}_1$	1.000	0.521	0.458	0.541	0.523	0.553	0.572	0.631	0.680	0.706	0.704
$\hat{\phi}_2$	1.000	0.465	0.523	0.465	0.480	0.453	0.437	0.369	0.320	0.295	0.293

3.2 Prediction, filtering and smoothing of time series

The issue of solving the already mentioned problems of prediction, filtering and smoothing will now be discussed. For these purposes, the already presented AR(p) model can be transformed into the form of a state-space model, for example, of the following form

$$\begin{pmatrix} y_t \\ y_{t-1} \\ \vdots \\ y_{t-p+2} \\ y_{t-p+1} \end{pmatrix} = \begin{pmatrix} \phi_1 & \phi_2 & \cdots & \phi_{p-1} & \phi_p \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{pmatrix} \begin{pmatrix} y_{t-1} \\ y_{t-2} \\ \vdots \\ y_{t-p+1} \\ y_{t-p} \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \\ 0 \end{pmatrix} \varepsilon_t,$$

$$y_t = (1 \quad 0 \quad \cdots \quad 0 \quad 0) \begin{pmatrix} y_t \\ y_{t-1} \\ \vdots \\ y_{t-p+2} \\ y_{t-p+1} \end{pmatrix}.$$

Also a more general ARMA(p, q) model

$$y_t = \phi_1 y_{t-1} + \phi_2 y_{t-2} + \cdots + \phi_p y_{t-p} + \varepsilon_t + \theta_1 \varepsilon_{t-1} + \theta_2 \varepsilon_{t-2} + \cdots + \theta_q \varepsilon_{t-q}.$$

can be transformed into the form of the state-space model. That can result, for example, in the following state equation and measurement equation [4, 8]

$$\boldsymbol{\alpha}_t = \begin{pmatrix} \phi_1 & 1 & 0 & \cdots & 0 \\ \phi_2 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \phi_{n-1} & 0 & \cdots & 0 & 1 \\ \phi_n & 0 & 0 & \cdots & 0 \end{pmatrix} \boldsymbol{\alpha}_{t-1} + \begin{pmatrix} 1 \\ \theta_1 \\ \vdots \\ \theta_{n-2} \\ \theta_{n-1} \end{pmatrix} \varepsilon_t,$$

$$y_t = (1 \quad 0 \quad \cdots \quad 0 \quad 0)\boldsymbol{\alpha}_t,$$

where

$$\boldsymbol{\alpha}_t = \begin{pmatrix} y_t \\ \phi_2 y_{t-1} + \cdots + \phi_n y_{t-n+1} + \theta_1 \varepsilon_t + \cdots + \theta_{n-1} \varepsilon_{t-n+2} \\ \vdots \\ \phi_n y_{t-1} + \theta_{n-1} \varepsilon_t \end{pmatrix}, \quad n = max(p, q+1),$$

$\phi_i = 0$ for $i > p$, and $\theta_i = 0$ for $i > q$.

However, it is necessary to mention the existence of a bigger number of alternative state-space models representing the same ARMA model; they differ from each other in their definitions of the state vector etc. These varied approaches are summarized in [10]. However, the state vectors defined in this way do not generally have a substantive interpretation.

The problem of prediction, filtering and smoothing will be demonstrated on a simulated ARMA(2,1) process

$$y_t = 0.6 y_{t-1} + 0.2 y_{t-2} + \varepsilon_t + 0.1 \varepsilon_{t-1}, \quad \varepsilon_t \sim N(0, 10), \quad t = 1, \ldots, 100.$$

In accordance with the above given, this ARMA model was transformed into the form of a state-space model. The problem of filtering was solved through a standard application of the algorithm of the Kalman filter (the prediction step and the correction step), as it was described in Section 2. The problem of prediction can be solved through applying merely the prediction step of the algorithm. This step is not followed by the correction step because the observation (measurement) which could improve the a priori estimate is not available yet (in the practical illustration, one-step ahead predictions were calculated in the observed period, and then predictions for five future times were calculated). The problem of smoothing can be solved in various ways. Here the application of the algorithm called the Rauch–Tung–Striebel smoother was used. In the first (forward) pass, the smoother applies the standard Kalman filter, and in the second (backward) pass, it processes recursively from the end, and by combining the filtered values, it computes smoothed values [15]. The obtained results are summarized in Fig. Fig**. 3**.

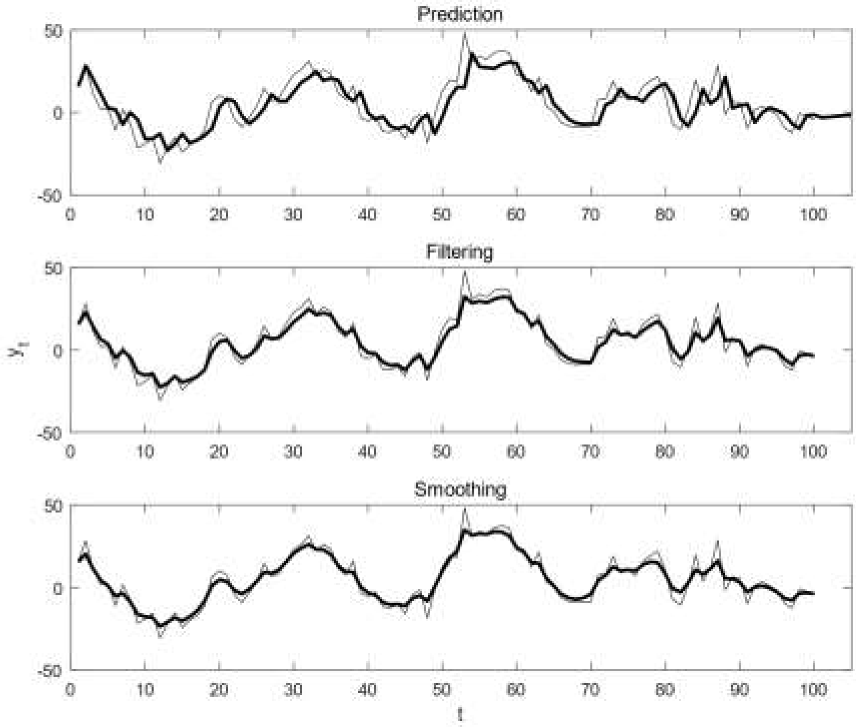

Fig. 3. Illustration of applying the Kalman filter for prediction, filtering and smoothing of a time series represented by ARMA(2,1) model.

4 Conclusion

Time series are essentially important in the sphere of dynamic models in economics. It is quite frequently necessary to estimate unobservable parameters of time series models, which can be done on the basis of observed values of economic variables. The paper presented some ways in which the Kalman filter can be used for estimating the AR(p) model´s parameters. A model of a simulated AR(2) process was used as a practical demonstration of this problem. From the results presented in Table Table **1** it is clear that the longer the observed series is, the better the estimates of the parameters of the model can be. Furthermore, using of the Kalman filter for solving problems of prediction, filtering and smoothing of time series was mentioned. The paper presented possible transformations of the AR(p) models or, more generally, the ARMA(p, q) models into the state-space model to which the Kalman filter can be applied. A model of a simulated ARMA(2,1) process was presented as a practical demonstration. The outputs were graphically illustrated in Fig. Fig. **3**, and it is clear from them that the results obtained from the application of the Kalman filter are suitable for simulated data.

The application of the Kalman filter in economics is convenient in the sphere of the estimation of the output gap of economic units [19, 1, 8], when the estimate of the position of the economic unit within the framework of the economic cycle is determined. Another convenient application can be in the field of financial estimations – to decide whether the currency policy of the given economy is restrictive or expansive [18] when determining the short-term or long-term interest rate. Our future research will focus on these issues and on mutual comparisons of the situation existing in the V4 countries.

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